# French Mathematical Olympiad 1996

Time: 5 hours.

1. Consider a triangles $ABC$ and points $D,E,F,G,H,I$ in the plane such that $ABED$, $BCGF$ and $ACHI$ are squares exterior to the triangle. Prove that points $D,E,F,G,H,I$ are concyclic if and only if one of the following two statements hold:

   (i) $ABC$ is an equilateral triangle.

   (ii) $ABC$ is an isosceles right triangle.

2. Let $a$ be an odd natural number and $b$ be a positive integer. We define a sequence of reals $(u_n)$ as follows: $u_0 = b$ and, for all $n \in \mathbb{N}_0$, $u_{n+1}$ is $u_n/2$ if $u_n$ is even and $a + u_n$ otherwise.

   (a) Prove that one can find an element of $u_n$ smaller than $a$.

   (b) Prove that the sequence is eventually periodic.

3. (a) Let be given a rectangular parallelepiped. Show that some four of its vertices determine a tetrahedron whose all faces are right triangles.

   (b) Conversely, prove that every tetrahedron whose all faces are right triangles can be obtained by selecting four vertices of a rectangular parallelepiped.

   (c) Now investigate such tetrahedra which also have at least two isosceles faces. Given the length $a$ of the shortest edge, compute the lengths of the other edges.

4. (a) A function $f$ is defined by $f(x) = x^x$ for all $x > 0$. Find the minimum value of $f$.

   (b) If $x$ and $y$ are two positive real numbers, show that $x^y + y^x > 1$.

5. Let $n$ be a positive integer. We say that a natural number $k$ has the property $C_n$ if there exist $2k$ distinct positive integers $a_1, b_1, \dots, a_k, b_k$ such that the sums $a_1 + b_1, \dots, a_k + b_k$ are distinct and strictly smaller than $n$.

   (a) Prove that if $k$ has the property $C_n$ then $k \leq \dfrac{2n-3}{5}$.

   (b) Prove that 5 has the property $C_{14}$.

   (c) If $\dfrac{2n-3}{5}$ is an integer, prove that it has the property $C_n$.