French Mathematical Olympiad 1995

Time: 5 hours.

1. We are given a triangle ABC in a plane P. To any line D, not parallel to any side of the triangle, we associate the barycenter G_D of the set of intersection points of D with the sides of $\triangle ABC$. The object of this problem is determining the set $\mathscr{F}$ of points G_D when D varies.

   (a) If D goes over all lines parallel to a given line δ, prove that G_D describes a line Δ_δ.

   (b) Assume $\triangle ABC$ is equilateral. Prove that all lines Δ_δ are tangent to the same circle as δ varies, and describe the set $\mathscr{F}$.

   (c) If ABC is an arbitrary triangle, prove that one can find a plane P and an equilateral triangle $A'B'C'$ whose orthogonal projection onto P is $\triangle ABC$, and describe the set $\mathscr{F}$ in the general case.

2. Study the convergence of a sequence defined by $u_0 \geq 0$ and $u_{n+1} = \sqrt{u_n} + \dfrac{1}{n+1}$ for all $n \in \mathbb{N}_0$.

3. Consider three circles in the plane $\Gamma_1, \Gamma_2, \Gamma_3$ of radii R passing through a point O, and denote by $\mathscr{D}$ the set of points of the plane which belong to at least two of these circles. Find the position of $\Gamma_1, \Gamma_2, \Gamma_3$ for which the area of $\mathscr{D}$ is minimum possible. Justify your answer.

4. Suppose $A_1, A_2, A_3, B_1, B_2, B_3$ are points in the plane such that for each $i, j \in \{1,2,3\}$ it holds that $A_iB_j = i + j$. What can be said about these six points?

5. Let f be a bijection from $\mathbb{N}$ into itself. Prove that one can always find three natural numbers a, b, c such that $a < b < c$ and $f(a) + f(c) = 2f(b)$.

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