French Mathematical Olympiad 1994

Time: 5 hours.

1. For each positive integer n, let I_n denote the number of integers p for which $50^n < 7^p < 50^{p+1}$.

   (a) Prove that, for each n, I_n is either 2 or 3.

   (b) Prove that $I_n = 3$ for infinitely many $n \in \mathbb{N}$, and find at least one such n.

2. Let be given a semi-sphere Σ whose base-circle lies on plane P. A variable plane Q, parallel to a fixed plane non-perpendicular to P, cuts Σ at a circle C. We denote by C' the orthogonal projection of C onto P. Find the position of Q for which the cylinder with bases C and C' has the maximum volume.

3. Let us define a function $f : \mathbb{N} \to \mathbb{N}_0$ by $f(1) = 0$ and, for all $n \in \mathbb{N}$,
$$f(2n) = 2f(n) + 1, \qquad f(2n+1) = 2f(n).$$
   Given a positive integer p, define a sequence (u_n) by $u_0 = p$ and $u_{k+1} = f(u_k)$ whenever $u_k \neq 0$.

   (a) Prove that, for each $p \in \mathbb{N}$, there is a unique integer $v(p)$ such that $u_{v(p)} = 0$.

   (b) Compute $v(1994)$. What is the smallest integer $p > 0$ for which $v(p) = v(1994)$?

   (c) Given an integer N, determine the smallest integer p such that $v(p) = N$.

4. Let ABC be a triangle. For any point P in the plane, let L, M, N be the feet of perpendiculars from P to sides BC, CA, AB respectively. Determine the point P for which $BL^2 + CM^2 + AN^2$ is minimal.

5. Assume $f : \mathbb{N} \to \mathbb{N}$ is a function such that $f(1) > 0$ and, for any natural numbers m and n,
$$f(m^2 + n^2) = f(m)^2 + f(n)^2.$$

   (a) Calculate $f(k)$ for $0 \leq k \leq 12$.

   (b) Calculate $f(n)$ for any natural number n.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com