French Mathematical Olympiad 1993

Time: 5 hours.

1. Assume we are given a set of weights, x_1 of which have mass d_1, x_2 have mass d_2, etc, x_k have mass d_k, where x_i, d_i are positive integers and $1 \le d_1 < d_2 < \cdots < d_k$. Let us denote their total sum by $n = x_1 d_1 + \cdots + x_k d_k$. We call such a set of weights *perfect* if each mass $0, 1, \ldots, n$ can be uniquely obtained using these weights.

   (a) Write down all sets of weights of total mass 5. Which of them are perfect?

   (b) Show that a perfect set of weights satisfies
   $$(1+x_1)(1+x_2)\ldots(1+x_k) = n+1.$$

   (c) Conversely, if $(1+x_1)(1+x_2)\ldots(1+x_k) = n+1$, prove that one can uniquely choose the corresponding masses $d_1, d_2, \ldots, d_k$ with $1 \le d_1 < \cdots < d_k$ in order that the obtained set of weights is perfect.

   (d) Determine all perfect sets of weights of total mass 1993.

2. Let n be a given positive integer.

   (a) Do there exist $2n+1$ consecutive positive integers $a_0, a_1, \ldots, a_{2n}$ in the ascending order such that $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n}$?

   (b) Do there exist consecutive positive integers $a_0, a_1, \ldots, a_{2n}$ in the ascending order such that $a_1^2 + \cdots + a_n^2 = a_{n+1}^2 + \cdots + a_{2n}^2$?

   (c) Do there exist consecutive positive integers $a_0, a_1, \ldots, a_{2n}$ in the ascending order such that $a_1^3 + \cdots + a_n^3 = a_{n+1}^3 + \cdots + a_{2n}^3$?
   You may study the function $f(x) = (x-n)^3 + \cdots + x^3 - (x+1)^3 - \cdots - (x+n)^3$ and prove that the equation $f(x) = 0$ has a unique solution x_n with $3n(n+1) < x_n < 3n(n+1)+1$. You may use the identity $1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/2$.

3. Let f be a function from $\mathbb{Z}$ to $\mathbb{R}$ which is bounded from above and satisfies $f(n) \le \frac{1}{2}\left(f(n-1) + f(n+1)\right)$ for all n. Show that f is constant.

4. We are given a disk $\mathscr{D}$ of radius 1 in the plane.

   (a) Prove that $\mathscr{D}$ cannot be covered with two disks of radii $r < 1$.

   (b) Prove that, for some $r < 1$, $\mathscr{D}$ can be covered with three disks of radii r. What is the smallest such r?

5. (a) Let be given two points A, B in the plane.

      i. Find the triangles MAB with a given area and the minimal perimeter.

      ii. Find the triangles MAB with a given parameter and the maximal area.

   (b) In a tetrahedron of volume V, let a, b, c, d be the lengths of its four edges, no three of which are coplanar, and let $L = a + b + c + d$. Determine the maximum value of V/L^3.

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