French Mathematical Olympiad 1992

Time: 5 hours.

1. Let Δ be a convex figure in a plane $\mathscr{P}$. Given a point $A \in \mathscr{P}$, to each pair (M,N) of points in Δ we associate the point $m \in \mathscr{P}$ such that $\overrightarrow{Am} = \overrightarrow{MN}/2$ and denote by $\delta_A(\Delta)$ the set of all so obtained points m.

   (a) i. Prove that $\delta_A(\Delta)$ is centrally symmetric.

   ii. Under which conditions is $\delta_A(\Delta) = \Delta$?

   iii. Let B,C be points in $\mathscr{P}$. Find a transformation which sends $\delta_B(\Delta)$ to $\delta_C(\Delta)$.

   (b) Determine $\delta_A(\Delta)$ if

   i. Δ is a set in the plane determined by two parallel lines.

   ii. Δ is bounded by a triangle.

   iii. Δ is a semi-disk.

   (c) Prove that in the cases $b.2$ and $b.3$ the lengths of the boundaries of Δ and $\delta_A(\Delta)$ are equal.

2. Let $\mathscr{C}$ be a circle of radius 1.

   (a) Determine the triangles ABC inscribed in $\mathscr{C}$ for which $AB^2 + BC^2 + CA^2$ is maximal.

   (b) Determine the quadrilaterals $ABCD$ inscribed in $\mathscr{C}$ for which $AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2$ is maximal.

3. Let $ABCD$ be a tetrahedron inscribed in a sphere with center O, and G and I be its barycenter and incenter respectively. Prove that the following are equivalent:

   (i) Points O and G coincide.

   (ii) The four faces of the terahedron are congruent.

   (iii) Points O and I coincide.

4. Given u_0,u_1 with $0 < u_0,u_1 < 1$, define the sequence (u_n) recurrently by the formula

$$u_{n+2} = \frac{1}{2}\left(\sqrt{u_{n+1}} + \sqrt{u_n}\right).$$

   (a) Prove that the sequence u_n is convergent and find its limit.

   (b) Prove that, starting from some index n_0, the sequence u_n is monotonous.

5. Determine the number of digits 1 in the integer part of $\dfrac{10^{1992}}{10^{83}+7}$.

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