French Mathematical Olympiad 1991

Time: 5 hours.

1. (a) Suppose that x_n ($n \geq 0$) is a sequence of real numbers with the property that $x_0^3 + x_1^3 + \cdots + x_n^3 = (x_0 + x_1 + \cdots + x_n)^2$ for each $n \in \mathbb{N}$. Prove that for each $n \in \mathbb{N}_0$ there exists $m \in \mathbb{N}_0$ such that $x_0 + x_1 + \cdots + x_n = \dfrac{m(m+1)}{2}$.

   (b) For natural numbers n and p, we define $S_{n,p} = 1^p + 2^p + \cdots + n^p$. Find all natural numbers p such that $S_{n,p}$ is a perfect square for each $n \in \mathbb{N}$.

2. For each $n \in \mathbb{N}$, the function f_n is defined on real numbers $x \geq n$ by

$$f_n(x) = \sqrt{x-n} + \sqrt{x-n+1} + \cdots + \sqrt{x+n} - (2n+1)\sqrt{n}.$$

   (a) If n is fixed, prove that $\lim_{x \to +\infty} f_n(x) = 0$.

   (b) Find the limit of $f_n(n)$ as $n \to +\infty$.

3. Let S be a fixed point on a sphere Σ with center Ω. Consider all tetrahedra $SABC$ inscribed in Σ such that SA, SB, SC are pairwise orthogonal.

   (a) Prove that all the planes ABC pass through a single point.

   (b) In one such tetrahedron, H and O are the orthogonal projections of S and Ω onto the plane ABC, respectively. Let R denote the circumradius of $\triangle ABC$. Prove that $R^2 = OH^2 + 2SH^2$.

4. Let p be a nonnegative integer and let $n = 2^p$. Consider all subsets A of the set $\{1, 2, \ldots, n\}$ with the property that, whenever $x \in A$, $2x \notin A$. Find the maximum number of elements that such a set A can have.

5. (a) For given complex numbers a_1, a_2, a_3, a_4, we define a function $P : \mathbb{C} \to \mathbb{C}$ by $P(z) = z^5 + a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z$. Let $w_k = e^{2ki\pi/5}$, where $k = 0, \ldots, 4$. Prove that

$$P(w_0) + P(w_1) + P(w_2) + P(w_3) + P(w_4) = 5.$$

   (b) Let A_1, A_2, A_3, A_4, A_5 be five points in the plane. A pentagon is inscribed in the circle with center A_1 and radius R. Prove that there is a vertex S of the pentagon for which

$$SA_1 \cdot SA_2 \cdot SA_3 \cdot SA_4 \cdot SA_5 \geq R^5.$$