French Mathematical Olympiad 1990

Time: 5 hours.

1. Let the sequence u_n be defined by $u_0 = 0$ and $u_{2n} = u_n$, $u_{2n+1} = 1 - u_n$ for each $n \in \mathbb{N}_0$.

   (a) Calculate u_{1990}.

   (b) Find the number of indices $n \leq 1990$ for which $u_n = 0$.

   (c) Let p be a natural number and $N = (2^p - 1)^2$. Find u_N.

2. A game consists of pieces of the shape of a regular tetrahedron of side 1. Each face of each piece is painted in one of n colors, and by this the faces of one piece are not necessarily painted in different colors. Determine the maximum possible number of pieces, no two of which are identical.

3. (a) Find all triples of integers (a, b, c) for which $\frac{1}{4} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$.

   (b) Determine all positive integers n for which there exist positive integers $x_1, x_2, \dots, x_n$ such that $1 = \frac{1}{x_1^2} + \frac{1}{x_2^2} + \cdots + \frac{1}{x_n^2}$.

4. (a) What is the maximum area of a triangle with vertices in a given square (or on its boundary)?

   (b) What is the maximum volume of a tetrahedron with vertices in a given cube (or on its boundary)?

5. In a triangle ABC, Γ denotes the excircle corresponding to A, A', B', C' are the points of tangency of Γ with BC, CA, AB respectively, and $S(ABC)$ denotes the region of the plane determined by segments AB', AC' and the arc $C'A'B'$ of Γ.

   Prove that there is a triangle ABC of a given perimeter p for which the area of $S(ABC)$ is maximum. For this triangle, give an approximate measure of the angle at A.

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