# French Mathematical Olympiad 1989

Time: 5 hours.

1. Given a figure $B$ in the plane, consider the figures $A$, containing $\mathscr{B}$, with the property *(P): a composition of an odd number of central symmetries with centers in A is also a central symmetry with center in A.* The task of this problem is to determine the smallest such figure, denoted by $\mathscr{A}$, that is contained in every figure $A$.

   (a) Determine the figure $\mathscr{A}$ if $B$ consists of: (1) two distinct points $I,J$; (2) three non-collinear points $I,J,K$.

   (b) Determine $\mathscr{A}$ if $B$ is a circle (with nonzero radius).

   (c) Give some examples of figures $B$ whose associated figures $\mathscr{A}$ are mutually distinct and distinct from the above ones.

2. (a) Let $z_1,z_2$ be complex numbers such that $z_1z_2 = 1$ and $|z_1 - z_2| = 2$. Let $A,B,M_1,M_2$ denote the points in complex plane corresponding to $-1,1,z_1,z_2$, respectively. Show that $AM_1BM_2$ is a trapezoid and compute the lengths of its non-parallel sides. Specify the particular cases.

   (b) Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be circles in the plane with centers $O_1$ and $O_2$, respectively, and with radius $d\sqrt{2}$, where $2d = O_1O_2$. Let $P$ and $Q$ be two variable points on $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively, both on $O_1O_2$ on on different sides of $O_1O_2$, such that $PQ = 2d$. Prove that the locus of midpoints $I$ of segments $PQ$ is the same as the locus of points $M$ with $MO_1 \cdot MO_2 = m$ for some $m$.

3. Find the greatest real $k$ such that, for every tetrahedron $ABCD$ of volume $V$, the product of areas of faces $ABC,ABD$ and $ACD$ is at least $kV^2$.

4. For natural numbers $x_1,\dots,x_k$, let $[x_k,\dots,x_1]$ be defined recurrently as follows: $[x_2,x_1] = x_2^{x_1}$ and, for $k \geq 3$, $[x_k,x_{k-1},\dots,x_1] = x_k^{[x_{k-1},\dots,x_1]}$.

   (a) Let $3 \leq a_1 \leq a_2 \leq \cdots \leq a_n$ be integers. For a permutation $\sigma$ of the set $\{1,2,\dots,n\}$, we set $P(\sigma) = [a_{\sigma(n)},a_{\sigma(n-1)},\dots,a_{\sigma(1)}]$. Find the permutations $\sigma$ for which $P(\sigma)$ is minimum or maximum.

   (b) Find all integers $a,b,c,d$, greater than or equal to 2, for which $[178,9] \leq [a,b,c,d] \leq [198,9]$.

5. Let $a_1,a_2,\dots,a_n$ be positive real numbers. Denote

$$s = \sum_{k=1}^{n} a_k \quad \text{and} \quad s' = \sum_{k=1}^{n} a_k^{1-1/k}.$$

   (a) Let $\lambda > 1$ be a real number. Show that $s' < \lambda s + \dfrac{\lambda}{\lambda - 1}$.

   (b) Deduce that $\sqrt{s'} < \sqrt{s} + 1$.



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