French Mathematical Olympiad 1988

Time: 5 hours.

1. Let us consider a matrix T with n rows denoted $1,\dots,n$ and p columns $1,\dots,p$. Its entries a_{ik} ($1 \le i \le n$, $1 \le k \le p$) are integers such that $1 \le a_{ik} \le N$, where N is a given natural number. Let E_i be the set of numbers that appear on the i-th row. Answer question (a) or (b).

   (a) Assume T satisfies the following conditions: (1) E_i has exactly p elements for each i, and (2) all E_i's are mutually distinct. Let m be the smallest value of N that permits a construction of such an $n \times p$ table T.

      i. Compute m if $n = p+1$.

      ii. Compute m if $n = 10^{30}$ and $p = 1988$.

      iii. Determine $\lim_{n\to\infty} \frac{m^p}{n}$, where p is fixed.

   (b) Assume T satisfies the following conditions instead: (1) $p = n$, (2) whenever i,k are integers with $i+k \le n$, the number a_{ik} is not in the set E_{i+k}.

      i. Prove that all E_i's are mutually different.

      ii. Prove that if $n \ge 2^q$ for some integer $q > 0$, then $N \ge q+1$.

      iii. Let $n = 2^r - 1$ for some integer $r > 0$. Prove that $N \ge r$ and show that there is such a table wih $N = r$.

2. For each $n \in \mathbb{N}$, determine the sign of $n^6 + 5n^5 \sin n + 1$.

   For which $n \in \mathbb{N}$ does it hold that $\dfrac{x^2 + 5n\cos n + 1}{n^6 + 5n^5 \sin n + 1} \ge 10^{-4}$.

3. Consider two spheres Σ_1 and Σ_2 and a line Δ not meeting them. Let C_i and r_i be the center and radius of Σ_i, and let H_i and d_i be the orthogonal projection of C_i onto Δ and the distance of C_i from Δ ($i = 1,2$). For a point M on Δ, let $\delta_i(M)$ be the length of a tangent MT_i to Σ_i, where $T_i \in \Sigma_i$ ($i = 1,2$). Find M on Δ for which $\delta_1(M) + \delta_2(M)$ is minimal.

4. A circle $\mathscr{C}$ and five distinct points M_1, M_2, M_3, M_4 and M on $\mathscr{C}$ are given in the plane. Prove that the product of the distances from M to lines M_1M_2 and M_3M_4 is equal to the product of the distances from M to the lines M_1M_3 and M_2M_4.

   What can one deduce for $2n+1$ distinct points $M_1, \dots, M_{2n}, M$ on $\mathscr{C}$?

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