French Mathematical Olympiad 1986

1. Let $ABCD$ be a tetrahedron.

   (a) Prove that the midpoints of the edges AB, AC, BD, and CD lie in a plane.

   (b) Find the point in that plane, whose sum of distances from the lines AD and BC is minimal.

2. Points A, B, C, and M are given in the plane.

   (a) Let D be the point in the plane such that $DA \leq CA$ and $DB \leq CB$. Prove that there exists point N satisfying $NA \leq MA$, $NB \leq MB$, and $ND \leq MC$.

   (b) Let A', B', C' be the points in the plane such that $A'B' \leq AB$, $A'C' \leq AC$, $B'C' \leq BC$. Does there exist point M' which satisfies the inequalities $M'A' \leq MA$, $M'B' \leq MB$, $M'C' \leq MC$?

3. (a) Prove or find a counter-example: For every two complex numbers z, w the following inequality holds:

   $$|z| + |w| \leq |z+w| + |z-w|.$$

   (b) Prove that for all $z_1, z_2, z_3, z_4 \in \mathbb{C}$:

   $$\sum_{k=1}^{4} |z_k| \leq \sum_{1 \leq i < j \leq 4} |z_i + z_j|.$$

4. For every sequence $\{a_n\}$ ($n \in \mathbb{N}$) we define the sequences $\{\Delta a_n\}$ and $\{\Delta^2 a_n\}$ by the following formulas:

   $$\begin{aligned} \Delta a_n &= a_{n+1} - a_n, \\ \Delta^2 a_n &= \Delta a_{n+1} - \Delta a_n. \end{aligned}$$

   Further, for all $n \in \mathbb{N}$ for which $\Delta a_n^2 \neq 0$, define

   $$a'_n = a_n - \frac{\Delta a_n)^2}{\Delta^2 a_n}.$$

   (a) For which sequences $\{a_n\}$ is the sequence $\{\Delta^2 a_n\}$ constant?

   (b) Find all sequences $\{a_n\}$, for which the numbers a'_n are defined for all $n \in \mathbb{N}$ and for which the sequence $\{a'_n\}$ is constant.

   (c) Assume that the sequence $\{a_n\}$ converges to $a = 0$, and $a_n \neq a$ for all $n \in \mathbb{N}$ and the sequence $\{\frac{a_{n+1}-a}{a_n-a}\}$ converges to $\lambda \neq 1$.

      i. Prove that $\lambda \in [-1, 1)$.

      ii. Prove that there exists $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have $\Delta^2 a_n \neq 0$.

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iii. Let $\lambda \neq 0$. For which $k \in \mathbb{Z}^+$ the sequence $\{\frac{a_n'}{a_{n+k}}$ is not convergent?

iv. Let $\lambda = 0$. Prove that the sequences $\{a_n'/a_n\}$ and $\{a_n'/a_{n+1}\}$ converge to 0. Find an example of $\{a_n\}$ for which the sequence $\{a_n'/a_{n+2}\}$ has a non-zero limit.

(d) What happens with part (c) if we remove the condition $a = 0$?

5. The functions $f, g : [0,1] \to \mathbb{R}$ are given with the formulas

$$f(x) = \sqrt[4]{1-x}, \;\; g(x) = f(f(x)),$$

and c denotes any solution of $x = f(x)$.

(a) i. Analyze the function $f(x)$ and draw its graph. Prove that the equation $f(x) = x$ has the unique root c satisfying $c \in [0.72, 0.73]$.

ii. Analyze the function $f'(x)$. Let M_1 and M_2 be the points of the graph of $f(x)$ with different x coordinates. What is the positin of the arc of M_1M_2 of the graph with respect to the segment M_1M_2?

iii. Analyze the function $g(x)$ and draw its graph. What is the position of that graph with respect to the line $y = x$? Find the tangents to the graph at points with x coordinates 0 and 1.

iv. Prove that every sequence $\{a_n\}$ with the conditions $a_1 \in (0,1)$ and $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$ converges. (consider the sequences $\{a_{2n-1}\}$, $\{a_{2n}\}$ ($n \in \mathbb{N}$ and teh function $g(x)$ associated with the graph).

(b) On the graph of the function $f(x)$ consider the points M and M' with x coordinates x and $f(x)$, where $x \neq c$.

i. Prove that the line MM' intersects with the line $y = x$ at point with x coordinate

$$h(x) = x - \frac{(f(x)-x)^2}{g(x) + x - 2f(x)}.$$

ii. Prove that if $x \in (0,c)$ then $h(x) \in (x,c)$.

iii. Analyze whether the sequence $\{a_n\}$ satisfying $a_1 \in (0,c)$, $a_{n+1} = h(a_n)$ for $n \in \mathbb{N}$ converges. Prove that the sequence $\left\{\frac{a_{n+1}-c}{a_n-c}\right\}$ converges and find its limit.

(c) Assume that the calculator approximates every number $b \in [-2,2]$ by number $\widetilde{b}$ having p decimal digits after the decimal point. We are performing the following sequence of operations on that calculator:

1) Set $a = 0.72$;
2) Calculate $\delta(a) = \widetilde{f(a)} - a$;
3) If $|\delta(a)| > 0.5 \cdot 10^{-p}$, then calculate $\widetilde{h(a)}$ and go to the operation 2) using $\widetilde{h(a)}$ instead of a;
4) If $|\delta(a)| \leq 0.5 \cdot 10^{-p}$, finish the calculation.

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Let $\bar{c}$ be the last of calculated values for $\widetilde{h(a)}$. Assuming that for each $x \in [0.72, 0.73]$ we have $|\widetilde{f(x)} - f(x)| < \varepsilon$, determine $\delta(\bar{c})$, the accuracy (depending on ε) of the approximation of c with $\bar{c}$.

(d) Assume that the sequence $\{a_n\}$ satisfies $a_1 = 0.72$ and $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. Find the smallest $n_0 \in \mathbb{N}$, such that for every $n \geq n_0$ we have $|a_n - c| < 10^{-6}$.