French Mathematical Competition 2003

Time: 5 hours.

Preliminary Part

(a) Let M be a point in the plane of a triangle ABC. Prove that

$$\overrightarrow{MA}\cdot\overrightarrow{BC}+\overrightarrow{MB}\cdot\overrightarrow{CA}+\overrightarrow{MC}\cdot\overrightarrow{AB}=0.$$

Deduce that the altitudes of triangle ABC are concurrent at a point H, called the orthocenter.

(b) Let Ω be the circumcenter of triangle ABC and H be the point given by $\overrightarrow{\Omega H}=\overrightarrow{\Omega A}+\overrightarrow{\Omega B}+\overrightarrow{\Omega C}$. Show that H is the orthocenter of $\triangle ABC$.

For every set X of points in the plane, we denote by $\mathscr{H}(X)$ the set of orthocenters of all triangles with the vertices in X. We call a planar set X *orthocentric* if it does not contain any line and contains the entire $\mathscr{H}(X)$.

Part 1.

(a) Find all orthocentric sets of three points.

(b) Find all orthocentric sets of four points.

(c) Let X be a set of four points on a circle and let $Y=\mathscr{H}(X)$.
   i. Prove that Y is the image of X under an isometry.
   ii. Determine $\mathscr{H}(Y)$.

(d) i. If Γ is a nondegenerate circle, determine $\mathscr{H}(\Gamma)$.
   ii. If D is a nondegenerate disc, determine $\mathscr{H}(D)$.

Part 2. In this part, R is a positive number, n an integer not smaller than 2, and X the set of vertices of a regular $2n$-gon inscribed in the circle with center O and radius R. Consider the set $\mathscr{T}$ of triangles with the vertices in X. An element of $\mathscr{T}$ is chosen at random.

(a) What is the probability of choosing a right-angled triangle?

(b) What is the probability of choosing an acute triangle?

(c) Let L be the squared distance from O to the orthocenter of the chosen triangle. Find the expected value of L.

Part 3.

(a) Let a,b,c be real numbers with $a(b-c)\neq 0$ and A,B,C be the points $(0,a)$, $(b,0)$, $(c,0)$, respectively. Compute the coordinates of the orthocenter D of $\triangle ABC$.

(b) Let X be the union of a line Δ and a point M outside Δ. Determine $\mathscr{H}(X)$. Prove that $\mathscr{H}(X)\cup X$ is an orthocentric set.

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(c) Let X be an orthocentric set contained in the union of the coordinate axes x and y and containing at least three points distinct from O.

i. Show that X contains at least three points on axis y with nonzero x-coordinates of the same sign.

ii. Show that X contains at least three points on axis y with positive x-coordinates.

(d) i. Find all finite orthocentric sets of at most five points contained in the union of the axes x and y.

ii. Let X be an orthocentric set contained in the union of the axes x and y and consisting of at least six points. Prove that there exist two sequences (x_n), (x'_n) of nonzero real numbers such that, for each n, the points $(x_n, 0)$ and $(x'_n, 0)$ are in X, and

$$\lim_{n\to\infty} x_n = \infty, \quad \lim_{n\to\infty} x'_n = 0.$$

Can the set X be finite?

Part 4. In this part we are concerned with constructing some remarkable orthocentric sets.

(a) Let k be a nonzero real number and Y be the hyperbola $xy = k$.

i. Let A, B, C, D be distinct points of Y, with the respective abscises a, b, c, d. Prove that AB and CD are orthogonal if and only if $abcd = -k^2$.

ii. With the above notation, find the orthocenter of triangle ABC.

iii. Prove that Y is orthocentric.

Throughout this part, q denotes a nonzero integer and X the set of points (x, y) satisfying $x^2 + qxy - y^2 = 1$.

(a) i. Prove that the equation $t^2 - qt - 1$ has two distinct real roots and show that these roots are irrational

Throughout this part, r and r' denote these roots and s the similitude defined by $z \mapsto (1 - ri)z$ in terms of complex numbers.

i. Prove that $s(X)$ is the hyperbola given by $xy = k$ for some real k, and find k. Deduce that X is an orthocentric set.

(b) Let G be the set of integer points of X and Γ be the set of x-coordinates of the elements of $s(G)$.

i. Show that Γ is the set of numbers of the form $x + ry$, where x, y are integers and $(x + ry)(x + r'y) = 1$.

ii. Show that $-1 \in \Gamma$ and $r^2 \in \Gamma$.

iii. Prove that a product of two elements of Γ and the inverse of an element of Γ are in Γ. Show that Γ is infinite.

(c) Conclude that the set G of integer points of X is an infinite orthocentric set.

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Part 5. Denote by Y_1 the hyperbola given by $xy = 1$ and by Y_0 the union of the coordinate axes. You may use the following result:

- Given points A, B, C, D in the plane, there exists a similitude s such that $s(A), s(B), s(C), s(D)$ all lie on Y_1 or all lie on Y_0.

Let A_0, B_0, C_0, D_0 be four points, no three collinear, and let $X_0 = \{A_0, B_0, C_0, D_0\}$. Define recurrently $X_{n+1} = \mathscr{H}(X_n)$ for $n \geq 0$. Suppose that there exists $n > 0$ for which $X_n = X_0$ and denote by m the smallest such n.

(a) Prove that $m = 1$ or $m = 2$.

(b) Find the sets X_0 for which $m = 1$ and those for which $m = 2$.

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