French Mathematical Competition 2002

Time: 5 hours.

Part 1. For a triangle ABC, we denote by P the orthogonal projection of A on BC and by D the reflection of C in AP.

Triangle ABC is said to be *pseudo-right* at A if $|\angle B - \angle C| = \frac{\pi}{2}$. Specially, it is pseudo-right at A and obtuse at B if $\angle B - \angle C = \frac{\pi}{2}$.

(a) Prove that triangle ABC is pseudo-right at A if and only if triangle ABD is right at A.

(b) Prove that $PA^2 = PB \cdot PC$ if and only if $\triangle ABC$ is right at A or pseudo-right at A.

(c) Prove that triangle ABC is pseudo-right at A if and only if its orthocenter is symmetric to A with respect to BC.

(d) Let R be the circumradius of $\triangle ABC$. Prove that $PB + PC = 2R$ if and only $\triangle ABC$ is right at A or pseudo-right at A.

(e) Prove that $\triangle ABC$ is pseudo-right at A if and only if the line AP is tangent to the circumcircle of $\triangle ABC$.

(f) Let α, β, γ be the points in the complex plane corresponding to A, B, C, respectively.

   i. Give a necessary and sufficient condition on $\dfrac{\alpha - \beta}{\alpha - \gamma}(\beta - \gamma)^2$ that $\triangle ABC$ is pseudo-right at A.

   ii. Set $\beta = -\gamma = e^{\frac{i\pi}{4}}$. Find the set E_1 of points A in the plane for which $\triangle ABC$ is pseudo-right at A.

   iii. Set $\beta = -\gamma = 1$. Find the set E_2 of points A in the plane for which $\triangle ABC$ is pseudo-right at A.

   iv. Which geometric transformation takes E_2 to E_1?

Part 2.

(a) Let (a, b, c) be a triple of positive numbers. Prove that the following conditions are equivalent:

   (i) There is a pseudo-right at A and obtuse at B triangle ABC with $AB = c$, $BC = a$, $CA = b$.

   (ii) $b^2 - c^2 = a\sqrt{b^2 + c^2}$.

   (iii) There exist real numbers $\rho > 0$ and $0 < \theta < \frac{\pi}{4}$ such that $a = \rho \cos 2\theta$, $b = \rho \cos\theta$, and $c = \rho \sin\theta$.

   If these conditions are satisfied, prove that θ is the measure of one of the angles of $\triangle ABC$. Can you give a geometric interpretation for ρ?

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(b) Let $\triangle ABC$ be pseudo-right at A and obtuse at B and let its side lengths be rational. Define ρ and θ as above. In this question you can use without proof that $\cos 2\varphi = \dfrac{1-\tan^2\varphi}{1+\tan^2\varphi}$ and $\sin 2\varphi = \dfrac{2\tan\varphi}{1+\tan^2\varphi}$

i. Prove that ρ is rational and deduce that so is $\tan\frac{\theta}{2}$. Let p,q be the coprime positive integers with $\tan\frac{\theta}{2} = \frac{p}{q}$.

ii. Prove that $0 < p < q(\sqrt{2}-1)$ and show the existence of positive rational number r such that

$$a = r(p^4 - 6p^2q^2 + q^4), \quad b = r(q^4 - r^4), \quad c = 2pqr(p^2+q^2).$$

(c) Conversely, show that the formulas in 2.(b) give side lengths of a triangle that is pseudo-right at A and obtuse at B.

(d) i. Let p and q be coprime positive integers. Find the greatest positive divisor of $p^4-6p^2q^2+q^4, q^4-p^4, 2pq(p^2+q^2)$ in terms of parity of p and q.

ii. Describe all triples of integers (a,b,c) for which there is a triangle ABC, pseudo-right at A and obtuse at B, with $AB=c$, $BC=a$, $CA=b$.

(e) Solve in $\mathbb{N}$ the equation $x^2(y^2+z^2) = (y^2-z^2)^2$.

(f) Solve in $\mathbb{Q}^*$ the equation $x^2(y^2+z^2) = (y^2-z^2)^2$.

(g) Solve in $\mathbb{N}$ the equation $x^2(y^2-z^2)^2 = (y^2+z^2)^3$.

Part 3. Let $\mathscr{H}$ be the curve defined by $x \ge 1$ and $y = \sqrt{x^2-1}$ and let $A=(r,s)$ be a point on $\mathscr{H}$. Denote by $\mathscr{A}$ the area of the set of points satisfying $1 \le x \le r$ and $y^2 \le x^2-1$.

(a) Calculate $\mathscr{A}$ in terms of r and s. (For example, you can rotate the image by $\frac{\pi}{4}$.)

(b) (Based on a result by Pierre Fermat in 1658.)

Let u be a positive and n be a natural number such that $u^n = r+s$. For each integer k, $1 \le k \le n$, consider the right-angled trapezoid (possibly degenerated into a triangle) having a lateral side with endpoints at $(u^{k-1},0)$ and $(u^k,0)$, the bases with slope -1, and the top right angle at the point on $\mathscr{H}$ with the abscise $\frac{u^{k-1}+u^{1-k}}{2}$.

i. Prove that the trapezoid T_k is well-defined for each k and draw a sketch.

ii. Why can we conjecture that the sum of the areas of these trapezoids has the limit $\frac{\mathscr{A}+s^2}{2}$ when u approaches $+\infty$?

iii. Prove the conjecture using another sequence of trapezoids combined with the first.

iv. Find the value of $\mathscr{A}$.

(c) Let $B=(1,0)$ and $C=(-1,0)$ and let $A=(x,y)$ be a point with $x,y \ge 0$ for which $\triangle ABC$ is pseudo-rectangle at A.

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Denote by S the area of $\triangle ABC$ and by S' the area of the part of the triangle consisting of points (X,Y) with $Y^2 \leq X^2 - 1$. Determine, if it exists, the limit of S'/S when $x \to \infty$.

Part 4. In the plane $z = 0$ in coordinate space, let $\mathscr{C}$ be the circle with center O and radius 1 and let T and P be distinct points such that TP is tangent to $\mathscr{C}$ at T. The line OP meets $\mathscr{C}$ at B and C, and $\mathscr{D}$ is the line through P perpendicular to the plane $z = 0$.

(a)
  i. Show that there exist two points A, A' on $\mathscr{D}$ such that triangles ABC and $A'BC$ are pseudo-right at A and A'. Show how to construct these points.
  ii. Prove that the coordinates of these two points satisfy $x^2 + y^2 = z^2 + 1$.

(b) Let $\mathscr{H}$ be the set of points A and A' when T and P vary.
  i. What is the intersection of $\mathscr{H}$ with a plane orthogonal to x-axis?
  ii. What is the intersection of $\mathscr{H}$ with a plane containing x-axis?
  iii. Prove that $\mathscr{H}$ is a union of lines and describe these lines.

(c) We are now interested in points of set $\mathscr{H}$ with integer coordinates.
  i. Let (x,y,z) be one such point. Prove that x or y is odd.
  Denote by $\mathscr{S}$ the set of points (x,y,z) with positive integer coordinates and with x odd such that $x^2 + y^2 = z^2 + 1$.
  ii. Let d be a fixed positive integer. Prove that the set of points $(x,y,z) \in \mathscr{S}$ with $\gcd(x+1, y+z)$ is empty if d is odd and infinite if d is even.
  iii. Let $m \geq 3$ be an integer. How many elements (x,y,z) of $\mathscr{S}$ with $x = m$ are there? Write down these elements for $m = 3, 5, 7, 9$.

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