French Mathematical Competition 2001

Time: 5 hours.

A *trio* is every triple (a,b,c) of nonzero real numbers satisfying $ab+bc+ca=0$. A trio is said to be *reduced* if $a+b+c=1$.

Part 1. We denote by C the set of points (x,y,z) in the coordinate space for which (x,y,z) is a trio, and by Γ the set of those for which (x,y,z) is a reduced trio. Let O be the origin and P be the plane given by $x+y+z=1$.

(a) Does there exist a trio (a,b,c) such that $a+b+c=0$?

(b) Prove that C is a union of lines passing through O, with O excluded.

(c) Prove that Γ is the intersection of a plane and a sphere with center O. Describe Γ geometrically

(d) Describe C geometrically and sketch it.

(e) Let L be a fixed point in Γ. If L' and L'' are arbitrary points on Γ, prove that the volume V of the tetrahedron $OLL'L''$ is maximal when the lines OL, OL', OL'' are orthogonal, and express the coordinates of L' and L'' in terms of those of L.

(f) Prove that the product abc attains its maximum and minimum values on Γ, and find the points at which those are attained.

Part 2. A trio (a,b,c) is called *rational* if a,b,c are rational, and *integer* if a,b,c are integers. We say that an integer trio is *primitive* if the greatest common divisor of a,b,c is 1.

(a) Describe the set H_1 of points $(x,y,1)$ such that $(a,b,1)$ is a trio. Show that the point $\Omega_1(-1,-1,1)$ is the center of symmetry of H_1. Find all points of H_1 with integer coordinates.

(b) For each nonzero integer h, denote by Z_h the set of integer trios (a,b,c) with $c=h$. Determine Z_h for $h=1$ and $h=2$.

(c) Prove that Z_h is a finite set and find the number $N(h)$ of its elements in terms of the number of divisors of h^2 in $\mathbb{Z}$. Prove that 4 divided $N(h)-2$.

(d) For every positive integer h, denote by $N'(h)$ the number of integer trios (a,b,c) such that at least one of a,b,c is equal to h. Express $N'(h)$ in terms of $N(h)$ depending on the parity of h.

(e) Prove that every integer trio (a,b,c) can be assigned a triple of integers (r,s,t) such that r and s are coprime, s is nonnegative, and

$$a=r(r+s)t, \quad b=s(r+s)t, \quad c=-rst.$$

State and verify the converse. For which trios (a,b,c) is not the triple (r,s,t) unique?

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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(f) Determine all triples (r,s,t) that are assigned to some primitive trios. Deduce that if (a,b,c) is a primitive trio, then $|abc|$, $|a+b|$, $|b+c|$ and $|c+a|$ are perfect squares.

(g) For each positive integer h, denote by $P(h)$ the number of primitive trios (a,b,c) with $c=h$. Prove that $P(h)$ is a power of 2. For which h is $P(h)=N(h)$? Give a sequence of integers (h_n) for which the sequence $P(h_n)/N(h_n)$ converges to zero.

(h) Let $(a,b,1)$ be a trio. Show that there exist sequences (x_n) and (y_n) converging respectively to a and b such that $(x_n,y_n,1)$ is a rational trio for all n.

(i) Let (a,b,c) be a reduced trio. Show that there exist sequences (x_n), (y_n) and (z_n) converging respectively to a,b and c such that (x_n,y_n,z_n) is a rational reduced trio for all n.

Part 3. Denote $j=e^{2i\pi/3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$. For each trio $T=(a,b,c)$ we define $\hat{T}=(a,c,b)$, $S(T)=a+b+c$ and $z(T)=a+bj+cj^2$.

(a) Express the module of $z(T)$ as a function of $S(T)$. Can we have $z(T)=0$? Compute the cosine and the sine of the argument θ of $z(T)$ in terms of a,b,c.

(b) Let z_0 be a given nonzero complex number. Find all trios $T=(a,b,c)$ such that $z(T)=z_0$.

(c) Given trios T_1 and T_2, prove that there is a unique trio, to be denoted as T_1*T_2, verifying $S(T_1*T_2)=S(T_1)S(T_2)$ and $z(T_1*T_2)=z(T_1)z(T_2)$. Compute T_1*T_2 in terms of T_1 and T_2. What can be said about the argument of $z(T_1*T_2)$? What can be said about that of $z(T_1*\hat{T}_1)$?

(d) If T_1 and T_2 are reduced trios, is T_1*T_2 so? The same question if the word "reduced" is replaced by "integer" and by "primitive".

(e) Compare the trios T_1*T_2 and T_2*T_1, $T_1*(T_2*T_3)$ and $T_1*(T_2*T_3)$, T_1 and $T_1*(1,0,0)$.

(f) Given trios T_1 and T_2, solve the equation $T_1*T=T_2$ in T.

(g) Given a trio T, define the sequence of trios (T_n) by $T_0=(1,0,0)$ and $T_{n+1}=T*T_n$. Calculate $S(T_n)$. Given an integer p, find all T for which $T_p=T_0$.

Part 4. Denote by A the set of integers m that are of the form u^2+3v^2, where u,v are integers. Denote by A' the set of nonzero complex numbers $z=u+iv\sqrt{3}$, where u,v are integers (note that $|z|^2=u^2+3v^2$). Denote by B the set of nonzero integers n of the form r^2+rs+s^2, where r,s are integers.

(a) Prove that a product of two elements of A' belongs to A', and that a product of two elements of A belongs to A.

(b) Show that if $p\in A$ is a prime number, then $p=3$ or $3\mid p-1$.

(c) Prove that $A=B$ (you may note that $r^2+rs+s^2=(r+s)^2-(r+s)s+s^2$).

The IMO Compendium Group,
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(d) Prove that every even element of A is divisible by 4 and that its quarter belongs to A; then prove that each element of A is the product of a power of 4 and an odd element of A.

(e) i. Suppose that there are an odd integer $m = u^2 + 3v^2$, where u, v are coprime integers, and a prime divisor p of m not belonging to A. Prove that there exists the smallest positive integer n_0 such that $n_0 p$ is in A, and that n_0 is odd.

ii. Verify the existence of integers u', v' less than $p/2$ in absolute value such that $u - u'$ and $v - v'$ are divisible by p. Prove that p divides the nonzero number $u'^2 + 3v'^2$ and hence that $n_0 < p$.

iii. Verify the existence of coprime nonzero integers u_0, v_0 such that $n_0 p = u_0^2 + 3v_0^2$.

iv. Verify the existence of integers u_1, v_1 less than $n_0/2$ in absolute value such that $u_1 - u_0$ and $v_1 - v_0$ are divisible by n. Prove that n_0 divides the nonzero integer $u_1^2 + 3v_1^2$ which we'll denote by $n_0 n_1$.

v. Deduce that such an integer m cannot exist (you may consider number $n_0^2 n_1 p$).

(f) Prove that every element of A can be written in the form $m = C^2 p_1 \cdots p_k$, where C is a positive integer and p_i distinct prime elements of A.

(g) i. Let p be a prime number with $3 \mid p - 1$, and K be the set of triples (x, y, z) of integers with $0 < x, y, z < p$ such that $p \mid xyz - 1$. Prove that K has exactly $(p-1)^2$ elements and that the number of those with x, y, z not all equal is divisible by 3.

ii. Deduce that there is an integer x with $1 < x < p$ such that p divides $x^2 + x + 1$, and then that p belongs to A. Describe the elements of A.

(h) Let D be the set of integers d for which there is an integer trio (a, b, c) satisfying $a + b + c = d$ and $abc \neq 0$. Prove, using question (e) of part 2, that every element of D has a prime divisor in A. Conversely, what can be said about the nonzero integers having a prime divisor in A?

(i) Find the elements of D between 2001 and 2010 inclusive.