French Mathematical Olympiad 2000

Time: 5 hours.

Exercise 1. We are given b white balls and n black balls ($b, n > 0$) which are to be distributed among two urns, at least one in each. Let s be the number of balls in the first urn, and r the number of white ones among them. One randomly chooses a urn and randomly picks a ball from it.

(a) Compute the probability p that the drawn ball is white.

(b) If s is fixed, for which r is p maximal?

(c) Find the distribution of balls among the urns which maximizes p.

(d) Give a generalization for larger numbers of colors and urns.

Problem. In this problem we consider so called *cartesian triangles*, that is, triangles ABC with integer sides $BC = a$, $CA = b$, $AB = c$ and $\angle A = 2\pi/3$. Unless noted otherwise, $\triangle ABC$ is assumed to be cartesian.

(a) If U, V, W are the projections of the orthocenter H to BC, CA, AB, respectively, specify those of the segments $AU, BV, CW, HA, HB, HC, HU, HV, HW, AW, AV, BU, BW, CV, CU$ having rational length.

(b) If I is the incenter, J the excenter across A, and P, Q the intersection points of the two bisectors at A with the line BC, specify those of the segments $PB, PC, QB, QC, AI, AJ, AP, AQ$ having rational length.

(c) Assume that b and c are prime. Prove that exactly one of the numbers $a+b-c$ and $a-b+c$ is a multiple of 3.

(d) Assume that $\frac{a+b-c}{3c} = \frac{p}{q}$, where p and q are coprime, and denote by d the gcd of $p(3p+2q)$ and $q(2p+q)$. Compute a, b, c in terms of p, q, d.

(e) Prove that if q is not a multiple of 3, then $d = 1$.

(f) Deduce a necessary and sufficient condition for a triangle to be cartesian with coprime integer sides, and by geometrical observations derive an analogous characterisation of triangles ABC with coprime sides $BC = a$, $CA = b$, $AB = c$ and $\angle A = \pi/3$.

Exercise 2. Let A, B, C be three distinct points in space, (A) the sphere with center A and radius r. Let E the set of numbers $R > 0$ for which there is a sphere (H) with center H and radius R such that B and C are outside the sphere, and the points of the sphere (A) are strictly inside it.

(a) Suppose that B and C are on a line with A and strictly outside (A). Show that E is nonempty and bounded, and determine its supremum in terms of the given data.

(b) Find a necessary and sufficient condition for E to be nonempty and bounded.

(c) Given r, compute the smallest possible supremum of E, if it exists.

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