Estonian IMO Team Selection Test 1997

Time: 4.5 hours each day.

First Day – Tartu, April 28

1. In a triangle ABC points A_1, B_1, C_1 are the midpoints of BC, CA, AB respectively, and A_2, B_2, C_2 are the midpoints of the altitudes from A, B, C respectively. Show that the lines A_1A_2, B_1B_2, C_1, C_2 are concurrent.

2. Prove that for all positive real numbers $a_1, a_2, \dots, a_n$,

$$\frac{1}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}}-\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\geq\frac{1}{n}.$$

   When does equality hold?

3. There are n boyfriend-girlfriend pairs at a party. Initially all the girls sit at a round table. For the first dance, each boy invites one of the girls to dance with. After each dance, a boy takes the girl he danced with to her seat, and for the next dance he invites the girl next to her in the counterclockwise direction. For which values of n can the girls be selected in such a way that in every dance at least one boy danced with his girlfriend, assuming that there are no less than n dances?

Second Day – Tartu, April 29

4. (a) Is it possible to partition the segment $[0,1]$ into two sets A and B and to define a continuous function f such that for every $x \in A$ $f(x)$ is in B, and for every $x \in B$ $f(x)$ is in A?

   (b) The same question with $[0,1]$ replaced by $[0,1)$.

5. A quadrilateral $ABCD$ is inscribed in a circle. On each of the sides AB, BC, CD, DA one erects a rectangle towards the interior of the quadrilateral, the other side of the rectangle being equal to CD, DA, AB, BC, respectively. Prove that the centers of these four rectangles are vertices of a rectangle.

6. It is known that for every integer $n > 1$ there is a prime number among the numbers $n+1, n+2, \dots, 2n-1$. Determine all positive integers n with the following property: Every integer $m > 1$ less than n and coprime to n is prime.

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