Estonian IMO Team Selection Test 1996

Time: 4.5 hours each day.

First Day – Tartu, April 13

1. Suppose that x, y and $\dfrac{x^2+y^2+6}{xy}$ are positive integers. Prove that $\dfrac{x^2+y^2+6}{xy}$ is a perfect cube.

2. Let a, b, c be the sides of a triangle, α, β, γ the corresponding angles and r the inradius. Prove that $a\sin\alpha + b\sin\beta + c\sin\gamma \geq 9r$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy for all x:

   (i) $f(x) = -f(-x)$;

   (ii) $f(x+1) = f(x)+1$;

   (iii) $f\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2} f(x)$ for $x \neq 0$.

Second Day – Tartu, April 14

4. Prove that the polynomial $P_n(x) = 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ has no real zeros if n is even and has exactly one real zero if n is odd.

5. Let H be the orthocenter of an obtuse triangle ABC and A_1, B_1, C_1 arbitrary points on the sides BC, CA, AB, respectively. Prove that the tangents drawn from H to the circles with diameters AA_1, BB_1, CC_1 are equal.

6. Each face of a cube is divided into n^2 equal squares. The vertices of the squares are called *nodes*, so each face has $(n+1)^2$ nodes.

   (a) If $n = 2$, does there exist a closed polygonal line whose links are sides of the squares and which passes through each node exactly once?

   (b) Prove that, for each n, such a polygonal line divides the surface area of the cube into two equal parts.

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