46-th Estonian Mathematical Olympiad 1999

Final Round – Tartu, March 11, 1999

Time allowed: 5 hours.

Grade 11

1. Find all pairs of integers (m,n) such that $(m-n)^2 = \dfrac{4mn}{m+n-1}$.

2. If $f(x) = \dfrac{x^2}{1+x^2}$, evaluate

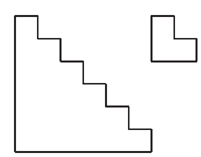
$$f\left(\frac{1}{2000}\right)+\cdots+f\left(\frac{1999}{2000}\right)+f\left(\frac{2000}{2000}\right)+f\left(\frac{2000}{1999}\right)+\cdots+f\left(\frac{2000}{1}\right).$$

3. Let ABC be a given triangle. Prove that a point X on the line AB satisfies

$$\overrightarrow{XA}\cdot\overrightarrow{XB}+\overrightarrow{XC}\cdot\overrightarrow{XC}=\overrightarrow{CA}\cdot\overrightarrow{CB}$$

if and only if X is the midpoint of AB or the feet of the altitude from C.

4. For which n can a sidewall with n equal stairs (the picture shows the case $n=6$) be tiled with tiles shown? A stair has the shape of a square of side 10dm, and a tile consists of three such squares.



5. On the fields $a1, a2, \dots, a8$ of a chessboard are put $2^0, 2^1, \dots, 2^7$ kernels of oat, on $b8, b7, \dots, b1$ are put $2^8, 2^9 \dots, 2^{15}$ kernels, on $c1, \dots, c8$ - $2^{16}, \dots, 2^{23}$ kernels, and so on (thus 2^{63} kernels are put on $h1$). A knight (horse in Estonian) starts moving on the chessboard from some field. Whenever it jumps to a certain field, it eats all the kernels of oat found on that field and continues moving, but then on that field the same number of kernels of oat grow again. After some time, the knight returns to the initial field (and eats the oat there). Prove that the number of kernels of oat eaten is divisible by 3.

Grade 12

1. Let a,b,c,d be nonnegative integers. Prove that the numbers 2^a7^b and 2^c7^d give the same remainder when divided by 15 if and only if the numbers 3^a5^b and 3^c5^d give the same remainder when divided by 16.

2. Evaluate the integral $\displaystyle\int_{-1}^{1} \ln\left(x+\sqrt{1+x^2}\right)\,dx$.

3. Show that in an acute-angled triangle the line joining the orthocenter and centroid is parallel to the line AB if and only if $\tan\angle A \cdot \tan\angle B = 3$.

4. Some checkers are arranged on the squares of a $2n \times 2n$ chessboard so that the number of checkers in each row and each column is odd. Prove that the number of checkers on the black squares is even.

5. Numbers $0, 1, \ldots, 9$ are arbitrarily arranged around a circle. Prove that:

   (a) there exist three consecutive numbers whose sum is at least 15;

   (b) there needn't exist three consecutive numbers whose sum is greater than 15.

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