45-th Estonian Mathematical Olympiad 1998

Final Round – Tartu, March 12, 1998

Time allowed: 5 hours.

Grade 11

1. Let d_1 and d_2 be divisors of a positive integer n. Suppose that the greatest common divisor of d_1 and n/d_2 and the greatest common divisor of d_2 and n/d_1 are equal. Show that $d_1 = d_2$.
2. In a triangle ABC, A_1, B_1, C_1 are the midpoints of segments BC, CA, AB, A_2, B_2, C_2 are the midpoints of segments B_1C_1, C_1A_1, A_1B_1, and A_3, B_3, C_3 are the incenters of triangles $B_1AC_1, C_1BA_1, A_1CB_1$, respectively. Show that the lines A_2A_3, B_2B_3 and C_2C_3 are concurrent.
3. A function f satisfies the conditions $f(x) \neq 0$ and $f(x+2) = f(x-1)f(x+5)$ for all real x. Show that $f(x+18) = f(x)$ for any real x.
4. A real number a satisfies the equality $\dfrac{1}{a} = a - [a]$. Prove that a is irrational.
5. A circle is divided into n equal arcs by n points. Assume that, no matter how we color the n points in two colors, there always exists an axis of symmetry of the set of points such that any two of the n points which are symmetric with respect to that axis have the same color. Find all possible values of n.

Grade 12

1. Solve the equation $x^2 + 1 = \log_2(x+2) - 2x$.
2. Find all prime numbers of the form $10101\dots01$.
3. In a triangle ABC, the bisector of the largest angle $\angle A$ meets BC at point D. Let E and F be the feet of perpendiculars from D to AC and AB, respectively. Let R denote the ratio between the areas of triangles DEB and DFC.
   (a) Prove that, for every real number $r > 0$, one can construct a triangle ABC for which R is equal to r.
   (b) Prove that if R is irrational, then at least one side length of $\triangle ABC$ is irrational.
   (c) Give an example of a triangle ABC with exactly two sides of irrational length, but with rational R.

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4. Find all integers $n > 2$ for which $(2n)! = (n-2)!n!(n+2)!$.

5. From an $n \times n$ square divided into n^2 unit squares, one corner unit square is cut off. Find all positive integers n for which it is possible to tile the remaining part of the square with L-trominos.

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