44-th Estonian Mathematical Olympiad 1997

Final Round – Tartu, April 3, 1997

Time allowed: 5 hours.

Grade 11

1. Prove that a positive integer n is composite if and only if there exist positive integers a,b,x,y such that $a+b=n$ and $\frac{x}{a}+\frac{y}{b}=1$.
2. Side lengths a,b,c of a triangle satisfy $\frac{a^3+b^3+c^3}{a+b+c}=c^2$. Find the measure of the angle opposite to side c.
3. Each diagonal of a convex pentagon is parallel to one of its sides. Prove that the ratio of the length of each diagonal to the length of the corresponding parallel side is the same, and find this ratio.
4. Let be given $n \geq 3$ distinct points in the plane. Is it always possible to find a circle which passes through three of the points and contains none of the remaining points
   (a) inside the circle;
   (b) inside the circle or on its boundary?
5. Six small circles of radius 1 are drawn so that they are all tangent to a larger circle, and two of them are tangent to sides BC and AD of a rectangle $ABCD$ at their midpoints K and L. Each of the remaining four small circles is tangent to two sides of the rectangle. The large circle is tangent to sides AB and CD of the rectangle and cuts the other two sides. Find the radius of the large circle.

Grade 12

1. For positive integers m and n we define
$$T(m,n)=\gcd\left(m,\frac{n}{\gcd(m,n)}\right).$$
   (a) Prove that there are infinitely many pairs (m,n) of positve integers for which $T(m,n)>1$ and $T(n,m)>1$.
   (b) Do there exist positive integers m,n such that $T(m,n)=T(n,m)>1$?

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2. A function f satisfies the following condition for each $n \in \mathbb{N}$:

$$f(1)+f(2)+\cdots+f(n)=n^2 f(n).$$

Find $f(1997)$ if $f(1)=999$.

3. A sphere is inscribed in a regular tetrahedron. Another regular tetrahedron is inscribed in the sphere. Find the ratio of the volumes of these two tetrahedra.

4. There are 19 lines in the plane dividing the plane into exactly 97 pieces.

   (a) Prove that among these pieces there is at least one triangle.

   (b) Show that it is indeed possible to place 19 lines in the above way.

5. Find the length of the longer side of the rectangle on the picture, if the shorter side has length 1 and the circles touch each other and the sides of the rectangle as shown.

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