43-th Estonian Mathematical Olympiad 1996

Final Round – Tartu, March 29, 1996

Time allowed: 5 hours.

Grade 11

1. Prove that for any positive numbers x, y it holds that $x^x y^y \geq x^y y^x$.
2. Three sides of a trapezoid are equal, and a circle with the longer base as a diameter halves the two non-parallel sides. Find the angles of the trapezoid.
3. Numbers $1992, 1993, \dots, 2000$ are written in a 3×3 table to form a magic square (i.e. the sums of numbers in rows, columns and big diagonals are all equal). Prove that the number in the center is 1996. Which numbers are placed in the corners?
4. Prove that, for each odd integer $n \geq 5$, the number $1^n + 2^n + \cdots + 15^n$ is divisible by 480.
5. Suppose that n triangles are given in the plane such that any three of them have a common vertex, but no four of them do. Find the greatest possible n.

Grade 12

1. Let p be a fixed prime. Find all pairs (x, y) of positive numbers satisfying $p(x - y) = xy$.
2. For which positive x does the expression
$$x^{1000} + x^{900} + x^{90} + x^6 + \frac{1996}{x}$$
attain the smallest value?
3. An equilateral triangle of side 1 is rotated around its center, yielding another equilarereal triangle. Find the area of the intersection of these two triangles.
4. Prove that for each prime number $p > 5$ there exists a positive integer n such that p^n ends in 001 in decimal representation.
5. Suppose that n teterahedra are given in space such that any two of them have at least two common vertices, but any three have at most one common vertex. Find the greatest possible n.

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