49-th Estonian Mathematical Olympiad 2002

Final Round – Tartu, March 7

Time allowed: 5 hours.

Grade 11

1. Find all real parameters a for which the equation $x^8 + ax^4 + 1 = 0$ has four real roots forming an arithmetic progression.

2. Inside an equilateral triangle there is a point whose distances from the sides of the triangle are 3, 4 and 5. Find the area of the triangle.

3. The teacher writes a 2002-digit number consisting only of digits 9 on the blackboard. The first student factors this number as ab with $a > 1$ and $b > 1$ and replaces it on the blackboard by two numbers a' and b' with $|a-a'| = |b-b'| = 2$. The second student chooses one of the numbers on the blackboard, factors it as cd with $c > 1$ and $d > 1$ and replaces the chosen number by two numbers c' and d' with $|c-c'| = |d-d'| = 2$, etc. Is it possible that after a certain number of students have been to the blackboard all numbers written there are equal to 9?

4. Let $a_1, \dots, a_5$ be real numbers such that at least N of the sums $a_i + a_j$ $(i < j)$ are integers. Find the greatest value of N for which it is possible that not all of the sums $a_i + a_j$ are integers.

5. Juku built a robot that moves along the border of a regular octagon, passing each side in exactly 1 minute. The robot starts in some vertex A and upon reaching each vertex can either continue in the same direction, or turn around and continue in the opposite direction. In how many different ways can the robot move so that after n minutes it will be in the vertex B opposite to A?

Grade 12

1. Peeter, J uri, Kati and Mari are standing at the entrance of a dark tunnel. They have one torch and none of them dares to be in the tunnel without it, but the tunnel is so narrow that at most two people can move together. It takes 1 minute for Peeter, 2 minutes for J uri, 5 for Kati and 10 for Mari to pass the tunnel. Find the minimum time in which they can all pass through the tunnel.

2. Does there exist an integer containing only digits 2 and 0 which is a k-th power of a positive integer ($k \geq 2$)?

3. Prove that for positive real numbers a, b and c the inequality
$$2(a^4 + b^4 + c^4) < (a^2 + b^2 + c^2)^2$$
holds if and only if a, b, c are the sides of a triangle.

4. A convex quadrilateral $ABCD$ is inscribed in a circle ω. The rays AD and BC meet in point K and the rays AB and DC meet in L. Prove that the circumcircle of triangle AKL is tangent to ω if and only if so is the circumcircle of triangle CKL.

5. There is a lottery at Juku's birthday party with a number of identical prizes, where each guest can win at most one prize. It is known that if there was one prize less, then the number of possible distributions of the prizes among the guests would be 50% less than it actually is, while if there was one prize more, then the number of possible distributions of the prizes would be 50% more than it actually is. Find the number of possible distributions of the prizes.

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