48-th Estonian Mathematical Olympiad 2001

Final Round – Tartu, March 29

Time allowed: 5 hours.

Grade 11

1. The angles of a convex n-gon are $\alpha, 2\alpha, \dots, n\alpha$. Find all possible values of n and the corresponding values of α.

2. A student wrote a correct addition operation $A/B + C/D = E/F$ on the blackboard, where both summands are irreducible and F is the least common multiple of B and D. After that, the student reduced the sum E/F correctly by an integer d. Prove that d is a common divisor of B and D.

3. Points D, E and F are taken on the sides BC, CA, AB of a triangle ABC respectively so that the segments AD, BE and CF intersect at point O. Prove that
$$\frac{AO}{OD} = \frac{AE}{EC} + \frac{AF}{FB}.$$

4. If x and y are nonnegative real numbers with $x + y = 2$, show that $x^2y^2(x^2 + y^2) \leq 2$.

5. Consider all trapezoids in a coordinate plane with interior angles of 90°, 90°, 45° and 135° whose bases are parallel to a coordinate axis and whose vertices have integer coordinates. Define the *size* of such a trapezoid as the total number of points with integer coordinates inside and on the boundary of the trapezoid.
   (a) How many pairwise non-congruent such trapezoids of size 2001 are there?
   (b) Find all positive integers not greater than 50 that do not appear as sizes of any such trapezoid.

Grade 12

1. Solve the system of equations
$$\sin x = y, \quad \sin y = x.$$

2. Find the maximum value of k for which one can choose k integers out of $1, 2, \dots, 2n$ so that none of them divides another one.

3. A circle with center I and radius r is inscribed in a triangle ABC with a right angle ay C. Rays AI and CI meet the opposite sides at D and E respectively. Prove that
$$\frac{1}{AE} + \frac{1}{BD} = \frac{1}{r}.$$

4. Prove that for any integer $a > 1$ there is a prime p for which $1 + a + a^2 + \cdots + a^{p-1}$ is composite.

5. A 3×3 table is filled with real numbers in such a way that each number in the table is equal to the absolute value of the difference of the sum of numbers in its row and the sum of numbers in its column.

   (a) Show that any number in this table can be expressed as a sum or a difference of some two numbers in the table.

   (b) Show that there is such a table not all of whose entries are 0.

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