48-th Czech and Slovak Mathematical Olympiad 1999

Category A

1. We are allowed to put several brackets in the expression

$$\frac{29 : 28 : 27 : 26 : \cdots : 17 : 16}{15 : 14 : 13 : 12 : \cdots : 3 : 2}.$$

   (a) Find the smallest possible integer value we can obtain in that way.

   (b) Find all possible integer values that can be obtained. *(J. Šimša)*

2. In a tetrahedron $ABCD$, E and F are the midpoints of the medians from A and D. Find the ratio of the volumes of tetrahedra $BCEF$ and $ABCD$.
(P. Leischner)

3. Show that there exists a triangle ABC such that $a \neq b$ and $a + t_a = b + t_b$, where t_a, t_b are the medians corresponding to a, b, respectively. Also prove that there exists a number k such that every such triangle satisfies $a + t_a = b + t_b = k(a+b)$. Finally, find all possible ratios $a : b$ in such triangles.
(J. Šimša)

4. In a certain language there are only two letters, A and B. We know that

   (i) There are no words of length 1, and the only words of length 2 are AB and BB.

   (ii) A segment of length $n > 2$ is a word if and only if it can be obtained from a word of length less than n by replacing each letter B by some (not necessarily the same) word.

   Prove that the number of words of length n is equal to $\dfrac{2^n + 2 \cdot (-1)^n}{3}$.
(P. Hliněný, P. Kaňovsk'y)

5. Given an acute angle APX in the plane, construct a square $ABCD$ such that P lies on the side BC and line PX meets CD in a point Q such that AP bisects the angle BAQ. *(J. Šimša)*

6. Find all pairs of real numbers a, b for which the system of equations

$$\frac{x+y}{x^2+y^2} = a, \qquad \frac{x^3+y^3}{x^2+y^2} = b$$

   has a real solution. *(J. Šimša)*

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