40-th Czech and Slovak Mathematical Olympiad 1991

1. Prove that for any real numbers p, q, r, φ,

$$p\cos^2\varphi + q\sin\varphi\cos\varphi + r\sin^2\varphi \geq \frac{1}{2}\left(p + r - \sqrt{(p-r)^2 + q^2}\right).$$

2. A museum has the shape of a (not necessarily convex) $3n$-gon. Prove that n custodians can be positioned so as to control all of the museum's space.

3. For any permutation p of the set $\{1, 2, \ldots, n\}$, let us denote

$$d(p) = |p(1) - 1| + |p(2) - 2| + \cdots + |p(n) - n|.$$

Let $i(p)$ be the number of inversions of p, i.e. the number of pairs $1 \leq i < j \leq n$ with $p(i) > p(j)$. Prove that $i(p) \leq d(p)$.

4. Prove that in all triangles ABC with $\angle A = 2\angle B$ the distance from C to A and to the perpendicular bisector of AB are in the same ratio.

5. In a group of mathematicians everybody has at least one friend (friendship is a symmetric relation). Show that there is a mathematician all of whose friends have average number of friends not smaller than the average number of friends in the whole group.

6. The set $\mathbb{N}$ is partitioned into three (disjoint) subsets A_1, A_2, A_3. Prove that at least one of them has the following property: There exists a positive number m such that for any k one can find numbers $a_1 < a_2 < \cdots < a_k$ in that subset satisfying $a_{j+1} - a_j \leq m$ for $j = 1, \ldots, k-1$.

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