56-th Czech and Slovak Mathematical Olympiad 2007

Third Round – March 18–21, 2007

Category A

1. A chess piece is placed on some square in an $n \times n$ square chessboard ($n \geq 2$). It then makes alternately *straight* and *diagonal* moves - i.e. moves to a square having a side or exactly one vertex in common with the original square, respectively. Find all n for which there exists a sequence of moves, starting by a diagonal move from the original square, such that the piece visits each square of the chessboard exactly once.

2. In a cyclic quadrilateral $ABCD$ denote by L and M the incenters of triangles BCA and BCD, respectively. The perpendiculars from L and M to the lines AC and BD respectively intersect at R. Show that the triangle LMR is isosceles.

3. Consider all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying $f(xf(y)) = yf(x)$ for any $x, y \in \mathbb{N}$. Find the least possible value of $f(2007)$.

4. The set M contains all natural numbers from 1 to 2007 inclusive and has the following property: If $n \in M$, then M contains all terms of the arithmetic progression with first term n and difference $n+1$. Decide whether there must always exist a number m such that M contains all natural numbers greater than m.

5. In an acute triangle ABC with $AC \neq BC$, points D and E are taken on sides BC and AC respectively such that $ABDE$ is a cyclic quadrilateral. The diagonals AD and BE meet at P. Show that if $CP \perp AB$ then P is the orthocenter of $\triangle ABC$.

6. Find all ordered triples (x, y, z) of mutually distinct real numbers satisfying the set equation
$$\{x, y, z\} = \left\{ \frac{x-y}{y-z}, \frac{y-z}{z-x}, \frac{z-x}{x-y} \right\}.$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com