55-th Czech and Slovak Mathematical Olympiad 2006

Third Round – Litoměřice, March 26–29, 2006

Category A

1. The sequence $(a_n)_{n=1}^{\infty}$ of positive integers satisfies $a_{n+1} = a_n + b_n$ for each $n \geq 1$, where b_n is obtained from a_n by reversing its digits (number b_n may start with zeros). For instance, $a_1 = 170$ yields $a_2 = 241$, $a_3 = 383$, $a_4 = 766$, etc. Decide whether a_7 can be a prime number. *(P. Novotný)*

2. Let m and n be natural numbers for which the equation
$$(x+m)(x+n) = x+m+n$$
has at least one integer solution. Show that $\frac{1}{2} < \frac{m}{n} < 2$. *(J. Šimša)*

3. In a non-equilateral triangle ABC, the angle bisectors at A and B meet the opposite sides at K and L, respectively. Moreover, S is the incenter, O the circumcenter, and V the orthocenter of the triangle. Prove that the following statements are equivalent:
   (a) Line KL is tangent to the circumcircles of the triangles ALS, BVS, and BKS.
   (b) Points A, B, K, L, O lie on a circle. *(T. Jurík)*

4. Segment AB is given on the plane. Find the locus of points C on the plane for which points A, B, orthocenter V, and incenter S of triangle ABC lie on a circle. *(J. Švrček)*

5. Find all triples (p,q,r) of distinct prime numbers having the following property:
$$p \mid q+r, \quad q \mid r+2p, \quad r \mid p+3q.$$
(M. Panák)

6. Solve in the real numbers the following system:
$$\begin{aligned} \tan^2 x + 2\cot^2 2y &= 1, \\ \tan^2 y + 2\cot^2 2z &= 1, \\ \tan^2 z + 2\cot^2 2x &= 1. \end{aligned}$$
(J. Švrček, P. Calábek)

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