52-nd Czech and Slovak Mathematical Olympiad 2003

Third Round – Liberec, March 30 - April 2, 2003

Category A

1. Solve the following system in the set of real numbers:

$$\begin{aligned} x^2 - xy + y^2 &= 7, \\ x^2y + xy^2 &= -2. \end{aligned}$$

(J. Földes)

2. On sides BC, CA, AB of a triangle ABC points D, E, F respectively are chosen so that AD, BE, CF have a common point, say G. Suppose that one can inscribe circles in the quadrilaterals $AFGE, BDGF, CEGD$ so that each two of them have a common point. Prove that triangle ABC is equilateral. *(M. Tancer)*

3. A sequence $(x_n)_{n=1}^{\infty}$ satisfies $x_1 = 1$ and for each $n > 1$,

$$x_n = \pm(n-1)x_{n-1} \pm (n-2)x_{n-2} \pm \cdots \pm 2x_2 \pm x_1.$$

Prove that the signs "$\pm$" can be chosen so that $x_n \neq 12$ holds only for finitely many n. *(P. Černek)*

4. Let be given an obtuse angle AKS in the plane. Construct a triangle ABC such that S is the midpoint of BC and K is the intersection point of BC with the bisector of $\angle BAC$. *(P. Leischner)*

5. Show that, for each integer $z \geq 3$, there exist two two-digit numbers A and B in base z, one equal to the other one read in reverse order, such that the equation $x^2 - Ax + B$ has one double root. Prove that this pair is unique for a given z. For instance, in base 10 these numbers are $A = 18$, $B = 81$. *(J. Šimša)*

6. If the product of positive numbers a, b, c equals 1, prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq a + b + c.$$

(P. Kaňovský)

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