51-st Czech and Slovak Mathematical Olympiad 2002

Third Round

Category A

1. Solve the following system in the set of integers:

$$\begin{aligned}(4x)_5+7y &= 14,\\ (2y)_5-(3x)_7 &= 74,\end{aligned}$$

where $(n)_k$ denotes the multiple of k closest to number n. *(P. Černek)*

2. Consider all equilateral triangles KLM with the property that K,L,M lie on the sides AB,BC,CD of a given square $ABCD$. Find the locus of the midpoint of the segment KL. *(J. Zhouf)*

3. Prove that a natural number A is a perfect square if and only if, for each $n\in\mathbb{N}$, at least one of the numbers $(A+1)^2-A,(A+2)^2-A,\dots,(A+n)^2-A$ is divisible by n. *(P. Kaňovský)*

4. Find all pairs of real numbers (a,b) for which the equation

$$\frac{ax^2-24x+b}{x^2-1}=x$$

has exactly two real solutions and their sum is 12. *(P. Černek)*

5. In a plane is given a triangle KLM and a point A on the extension of side KL over K. Construct a rectangle $ABCD$ whose vertices B,C and D lie on lines KM,KL and LM respectively. *(P. Calábek)*

6. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that for all $x,y\in\mathbb{R}^+$,

$$f(xf(y))=f(xy)+x. \qquad \textit{(P. Kaňovský)}$$

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