50-th Czech and Slovak Mathematical Olympiad 2001

Third Round – Praha, April 1-4, 2001

Category A

1. Determine all polynomials P such that for every real number x,
$$P(x)^2+P(-x)=P(x^2)+P(x).$$ *(P. Calábek)*

2. Given a triangle PQX in the plane, with $PQ=3$, $PX=2.6$ and $QX=3.8$. Construct a right-angled triangle ABC such that the incircle of $\triangle ABC$ touches AB at P and BC at Q, and point X lies on the line AC. *(J. Šimša)*

3. Find all triples of real numbers (a,b,c) for which the set of solutions x of
$$\sqrt{2x^2+ax+b}>x-c$$
is the set $(-\infty,0)\cup(1,\infty)$. *(P. Černek)*

4. In a certain language there are n letters. A sequence of letters is a *word*, if there are no two equal letters between two other equal letters. Find the number of words of the maximum length. *(K. Černeková)*

5. A sheet of paper has the shape of an isosceles trapezoid $C_1AB_2C_2$ with the shorter base B_2C_2. The foot of the perpendicular from the midpoint D of C_1C_2 to AC_1 is denoted by B_1. Suppose that upon folding the paper along DB_1, AD and AC_1 points C_1,C_2 become a single point C and points B_1,B_2 become a point B. The area of the tetrahedron $ABCD$ is 64cm^2. Find the sides of the initial trapezoid. *(P. Leischner)*

6. Let be given natural numbers $a_1,a_2,\dots,a_n$ and a function $f:\mathbb{Z}\to\mathbb{R}$ such that $f(x)=1$ for all integers $x<0$ and
$$f(x)=1-f(x-a_1)f(x-a_2)\cdots f(x-a_n)$$
for all integers $x\geq 0$. Prove that there exist natural numbers s and t such that for all integers $x>s$ it holds that $f(x+t)=f(x)$. *(P. Kaňovský)*

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com