7-th Croatian National Mathematical Competition 1998

High School

Kraljevica, May 7–10, 1998

1-st Grade

1. Which number is greater:

$$A = \frac{2.00\dots04}{1.00\dots04^2 + 2.00\dots04}, \quad \text{or} \quad B = \frac{2.00\dots02}{1.00\dots02^2 + 2.00\dots02},$$

   where each of the numbers above contains 1998 zeros?

2. Find all positive integer solutions of the equation $10(m+n) = mn$.

3. Ivan and Krešo started to travel from Crikvenica to Kraljevica, whose distance is 15 km, and at the same time Marko started from Kraljevica to Crikvenica. Each of them can go either walking with the speed of 5 km/h, or by bicycle with the speed of 15 km/h. Ivan started walking, and Krešo was driving a bicycle until meeting Marko. Then Krešo gave the bicycle to Marko and continued walking to Kraljevica, while Marko continued to Crikvenica by bycicle. When Marko met Ivan, he gave him the bicycle and continued by foot, so Ivan arrived to Kraljevica by bicycle. Find, for each of them, the time he spent in travel as well as the time spent in walking.

4. Let be given a regular hexagon of side length 1. Six circles with the sides of the hexagon as diameters are drawn. Find the area of the part of the hexagon lying outside all the circles.

2-nd Grade

1. Solve the equation $2z^3 - (5+6i)z^2 + 9iz + 1 - 3i = 0$, knowing that one of the solutions is real.

2. If a, b are nonnegative real numbers, prove the inequality

$$\frac{a + \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + b}{4} \leq \frac{a + \sqrt{ab} + b}{3}.$$

3. Points E and F are chosen on the sides AB and BC respectively of a square $ABCD$ such that $BE = BF$. Let BN be an altitude of the triangle BCE. Prove that the triangle DNF is right-angled.

4. For natural numbers m, n, set $a = (n+1)^m - n$ and $b = (n+1)^{m+3} - n$.

   (a) Prove that a and b are coprime if m is not divisible by 3.

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(b) Find all numbers m, n for which a and b are not coprime.

3-rd Grade

1. Let a, b, c be the sides and α, β, γ be the corresponding angles of a triangle. Prove the equality

$$\left(\frac{b}{c}+\frac{c}{b}\right)\cos\alpha+\left(\frac{c}{a}+\frac{a}{c}\right)\cos\beta+\left(\frac{a}{b}+\frac{b}{a}\right)\cos\gamma=3.$$

2. A hemisphere is inscribed in a cone so that its base lies on the base of the cone. The ratio of the area of the entire surface of the cone to the area of the hemisphere (without the base) is $18/5$. Compute the angle at the vertex of the cone.

3. Let AA_1, BB_1, CC_1 be the altitudes of a triangle ABC. If $\overrightarrow{AA_1}+\overrightarrow{BB_1}+\overrightarrow{CC_1}=0$, prove that the triangle ABC is equilateral.

4. Prove that among any 79 consecutive natural numbers there exists one whose sum of digits is divisible by 13. Find a sequence of 78 consecutive natural numbers for which the above statement fails.

4-th Grade

1. Let be given a parabola $y^2 = 4ax$ in the coordinate plane. Consider all chords of the parabola that are visible at a right angle from the origin of the coordinate system. Prove that all these chords pass through a fixed point.

2. Let a and m be positive integers and p be an odd prime number such that $p^m \mid a-1$ and $p^{m+1} \nmid a-1$. Prove that

   (a) $p^{m+n} \mid a^{p^n}-1$ for all $n \in \mathbb{N}$, and

   (b) $p^{m+n+1} \nmid a^{p^n}-1$ for all $n \in \mathbb{N}$.

3. Let $A = \{1, 2, \dots, 2n\}$ and let the function $g : A \to A$ be defined by $g(k) = 2n-k+1$. Does there exist a function $f : A \to A$ such that $f(k) \neq g(k)$ and $f(f(f(k))) = g(k)$ for all $k \in A$, if (a) $n = 999$; (b) $n = 1000$?

4. Eight bulbs are arranged on a circle. In every step we perform the following operation: We simultaneously switch off all those bulbs whose two neighboring bulbs are in different states, and switch on the other bulbs. Prove that after at most four steps all the bulbs will be switched on.