6-th Croatian National Mathematical Competition 1997

High School

Novi Vinodolski, May 8–11, 1997

1-st Grade

1. Let n be a natural number. Solve the equation
$$||\cdots|||x-1|-2|-3|-\cdots-(n-1)|-n|=0.$$

2. Given are real numbers $a<b<c<d$. Determine all permutations p,q,r,s of the numbers a,b,c,d for which the value of the sum
$$(p-q)^2+(q-r)^2+(r-s)^2+(s-p)^2$$
is minimal.

3. A chord divides the interior of a circle k into two parts. Variable circles k_1 and k_2 are inscribed in these two parts, touching the chord in the same point. Show that the ratio of the radii of circles k_1 and k_2 is constant, i.e. independent of the tangency point with the chord.

4. An infinite sheet of paper is divided into equal squares, some of which are colored red. In each 2×3 rectangle there are exactly two red squares. Now consider an arbitrary 9×11 rectangle. How many red squares does it contain? (The sides of all considered rectangles go along the grid lines.)

2-nd Grade

1. In a regular hexagon $ABCDEF$ with center O, points M and N are the midpoints of the sides CD and DE, and L the intersection point of AM and BN. Prove that:
   (a) ABL and $DMLN$ have equal areas;
   (b) $\angle ALD=\angle OLN=60^\circ$;
   (c) $\angle OLD=90^\circ$.

2. For any different positive numbers a,b,c prove the inequality
$$a^ab^bc^c>a^bb^cc^a.$$

3. Number 2^{1997} has m decimal digits, while number 5^{1997} has n digits. Evaluate $m+n$.

4. In the plane are given 1997 points. Show that among the pairwise distances between these points there are at least 32 different values.

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3-rd Grade

1. Integers x, y, z and a, b, c satisfy

$$x^2 + y^2 = a^2, \quad y^2 + z^2 = b^2, \quad z^2 + x^2 = c^2.$$

Prove that the product xyz is divisible by (a) 5, and (b) 55.

2. Prove that for every real number x and positive integer n

$$|\cos x| + |\cos 2x| + |\cos 2^2 x| + \cdots + |\cos 2^n x| \geq \frac{n}{2\sqrt{2}}.$$

3. The areas of the faces ABD, ACD, BCD, BCA of a tetrahedron $ABCD$ are S_1, S_2, Q_1, Q_2, respectively. The angle between the faces ABD and ACD equals α, and the angle between BCD and BCA is β. Prove that

$$S_1^2 + S_2^2 - 2S_1S_2\cos\alpha = Q_1^2 + Q_2^2 - 2Q_1Q_2\cos\beta.$$

4. On the sides of a triangle ABC are constructed similar triangles ABD, BCE, CAF with $k = AD/DB = BE/EC = CF/FA$ and $\alpha = \angle ADB = \angle BEC = \angle CFA$. Prove that the midpoints of the segments AC, BC, CD and EF form a parallelogram with an angle α and two sides whose ratio is k.

4-th Grade

1. Find the last four digits of each of the numbers 3^{1000} and 3^{1997}.

2. Consider a circle k and a point K in the plane. For any two distinct points P and Q on k, denote by k' the circle through P, Q and K. The tangent to k' at K meets the line PQ at point M. Describe the locus of the points M when P and Q assume all possible positions.

3. Function f is defined on the positive integers by $f(1) = 1$, $f(2) = 2$ and

$$f(n+2) = f(n+2-f(n+1)) + f(n+1-f(n)) \quad \text{for } n \geq 1.$$

   (a) Prove that $f(n+1) - f(n) \in \{0, 1\}$ for each $n \geq 1$.
   (b) Show that if $f(n)$ is odd then $f(n+1) = f(n) + 1$.
   (c) For each positive integer k find all n for which $f(n) = 2^{k-1} + 1$.

4. Let k be a natural number. Determine the number of non-congruent triangles with the vertices at vertices of a given regular $6k$-gon.

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