5-th Croatian National Mathematical Competition 1996

High School

Kraljevica, May 16–19, 1996

1-st Grade

1. Prove that $a^4 - 10a^2 + 9$ is divisible by 1920 for every prime number $a > 5$.

2. Real numbers a, b, c, d satisfy the condition $a + b + c + d = 0$. Let us denote $S_1 = ab + bc + cd$ and $S_2 = ac + ad + bd$. Prove that
$$5S_1 + 8S_2 \leq 0 \quad \text{and} \quad 8S_1 + 5S_2 \leq 0.$$

3. In a convex pentagon $ABCDE$, M, N, P, Q are the midpoints of AB, BC, CD, DE respectively, and R and S are the midpoints of MP and QN. Prove that $SR = \frac{1}{4}AE$.

4. Four circles of radius a with centers at the vertices of a square with side a divide the square into nine regions. Compute the area of each of the regions in terms of the area Q of the square, the area K of any of the circles, and the area T of an equilateral triangle with side a.

2-nd Grade

1. If a function f satisfies the conditions (i)–(iii), determine $f\left(\sqrt{1996}\right)$.

   (i) $f(1) = 1$;

   (ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$;

   (iii) $f(1/x) = f(x)/x^2$ for all $x \in \mathbb{R}$, $x \neq 0$.

2. For which real nhumbers a, b are the modules of all the roots of the equation $z^3 + az^2 + bz - 1$ equal to 1?

3. Let S be the intersection of the diagonals of a convex quadrilateral $A_1A_2A_3A_4$. Denote by s_k the area of the triangle A_kSA_{k+1}, $k = 1, 2, 3, 4$ (where $A_5 = A_1$). Prove that $s_2^2 = s_1s_3$ and $2s_4 = s_1 + s_3$ if and only if $A_1A_2A_3A_4$ is a parallelogram.

4. In a circle k of radius R, OA is a diameter and OB a chord. The tangent to k at A intersects the line OB at C, and T is a point on the segment OB such that $OT = BC$. If T' is the projection of T onto OA, express TT' in terms of $x = OT'$.

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3-rd Grade

1. Prove that for all $x \in \mathbb{R}$, $\sin^5 x + \cos^5 x + \sin^4 x \leq 2$. When does equality occur?

2. Let h_1, h_2, h_3 be the altitides of a triangle ABC from A, B, C respectively, and let u, v, w be the distances of a point M inside the triangle from the sides BC, CA, AB, respectively. Prove that
$$\frac{h_1}{u} + \frac{h_2}{v} + \frac{h_3}{w} \geq 9, \quad h_1h_2h_3 \geq 27uvw, \quad (h_1 - u)(h_2 - v)(h_3 - w) \geq 8uvw.$$

3. A regular quadrilateral pyramid is cut by a plane passing through one of the vertices of the base and is perpendicular to the opposite lateral edge. The area of the intersection is half the area of the base. Determine the angle between a lateral edge and the base.

4. Let α and β be positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Consider $A = \{[n\alpha] \mid n \in \mathbb{N}\}$ and $B = \{[n\beta] \mid n \in \mathbb{N}\}$. Show that $A \cup B = \mathbb{N}$ and $A \cap B = \emptyset$.
Hint: You may prove an equivalent statement: $\phi(m) = m$ for $m \in \mathbb{N}$, where $\phi(m) = \#\{k \in \mathbb{N} \cap A \mid k \leq m\} + \#\{k \in \mathbb{N} \cap B \mid k \leq m\}$.

4-th Grade

1. Does the following equation have a solution:
$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345?$$

2. For which real values of λ_1, λ_2 are all solutions to the equation
$$(x + i\lambda_1)^n + (x + i\lambda_2)^n = 0$$
real? Determine these solutions.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that are continuous at zero and satisfy the condition
$$f(x) - 2f(tx) + f(t^2x) = x^2 \quad \text{for all } x \in \mathbb{R},$$
where $t \in (0, 1)$ is a given number.

4. *Problem 4 for Grade 3.*

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