10-th Croatian National Mathematical Competition 2001

High School

Makarska, May 9–12, 2001

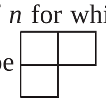
1-st Grade

1. Find all integers x for which $2x^2 - x - 36$ is a square of a prime number.

2. Let S be the center of a square $ABCD$ and P be the midpoint of AB. The lines AC and PD meet at M, and the lines BD and PC meet at N. Prove that the radius of the incircle of the quadrilateral $PMSN$ equals $MP - MS$.

3. Let a and b be positive real numbers. Prove the inequality

$$\sqrt[3]{\frac{a}{b}} + \sqrt[3]{\frac{b}{a}} \leq \sqrt[3]{2(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)}.$$

4. Find all possible values of n for which a rectangular board $9 \times n$ can be partitioned into tiles of the shape 

2-nd Grade

1. Let $z \neq 0$ be a complex number such that $z^8 = \bar{z}$. What are the possible values of z^{2001}?

2. The excircle of a triangle ABC corresponding to A touches the side BC at K and the rays AB and AC at P and Q, respectively. The lines OB and OC intersect PQ at M and N, respectively. Prove that

$$\frac{QN}{AB} = \frac{NM}{BC} = \frac{MP}{CA}.$$

3. Let be given triples of integers (r_j, s_j, t_j), $j = 1, 2, \dots, N$, such that for each j, r_j, s_j, t_j are not all even. Show that one can find integers a, b, c such that $ar_j + bs_j + ct_j$ is odd for at least $4N/7$ of the indices j.

4. On the coordinate plane is given a polygon $\mathscr{P}$ of area greater than 1. Prove that there exist two different points (x_1, y_1) and (x_2, y_2) inside the polygon $\mathscr{P}$ such that $x_1 - x_2$ and $y_1 - y_2$ are both integers.

3-rd Grade

1. Let O and P be fixed points on a plane, and let $ABCD$ be any parallelogram with center O. Let M and N be the midpoints of AP and BP respectively. Lines MC and ND meet at Q. Prove that the point Q lies on the line OP, and show that it is independent on the choice of the parallelogram $ABCD$.

2. In a triangle ABC with $AC \neq BC$, M is the midpoint of AB and $\angle A = \alpha$, $\angle B = \beta$, $\angle ACM = \varphi$ and $\angle BCM = \psi$. Prove that
$$\frac{\sin\alpha\sin\beta}{\sin(\alpha-\beta)} = \frac{\sin\varphi\sin\psi}{\sin(\varphi-\psi)}.$$

3. Numbers $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{2001}$ are written on a blackboard. A student erases two numbers x, y and writes down the number $x + y + xy$ instead. Determine the number that will be written on the board after 2000 such operations.

4. Let S be a set of 100 positive integers less than 200. Prove that there exists a nonempty subset T of S the product of whose elements is a perfect square.

4-th Grade

1. On the unit circle k with center O, points A and B with $AB = 1$ are chosen and unit circles k_1 and k_2 with centers A and B are drawn. A sequence of circles (l_n) is defined as follows: circle l_1 is tangent to k internally at D_1 and to k_1, k_2 externally, and for $n > 1$ circle l_n is tangent to k_1 and k_2 and to l_{n-1} at D_n. For each n, compute $d_n = OD_n$ and the radius r_n of l_n.

2. A piece of paper in the shape of a square $FBHD$ with side a is given. Points G, A on FB and E, C on BH are marked so that $FG = GA = AB$ and $BE = EC = CH$. The paper is folded along DG, DA, DC and AC so that G overlaps with B, and F and H overlap with E. Compute the volume of the obtained tetrahedron $ABCD$.

3. Let p_1, p_2, p_3, p_4 be four distinct primes, and let $1 = d_1 < d_2 < \cdots < d_{16} = n$ be the divisors of $n = p_1p_2p_3p_4$. Determine all $n < 2001$ with the property that $d_9 - d_8 = 22$.

4. Suppose that zeros and ones are written in the cells of an $n \times n$ board, in such a way that the four cells in the intersection of any two rows and any two columns contain at least one zero. Prove that the number of ones does not exceed $\frac{n}{2}\left(1+\sqrt{4n-3}\right)$.

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