9-th Croatian National Mathematical Competition 2000

High School

Mali Lošinj, May 10–13, 2000

1-st Grade

1. Find all positive integer solutions of the equation $\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=1$.
2. The incircle of a triangle ABC touches BC, CA, AB at A_1, B_1, C_1, respectively. Find the angles of $\triangle A_1B_1C_1$ in terms of the angles of $\triangle ABC$.
3. Let $m>1$ be an integer. Determine the number of positive integer solutions of the equation $\left[\frac{x}{m}\right]=\left[\frac{x}{m-1}\right]$.
4. We are given coins of $1,2,5,10,20,50$ lipas and of 1 kuna (Croatian currency: 1 kuna = 100 lipas). Prove that if a bill of M lipas can be paid by N coins, then a bill of N kunas can be paid by M coins.

2-nd Grade

1. Let $a>0$ and x_1,x_2,x_3 be real numbers with $x_1+x_2+x_3=0$. Prove that
$$\log_2(1+a^{x_1})+\log_2(1+a^{x_2})+\log_2(1+a^{x_3})\geq 3.$$
2. Two squares $ACXE$ and $CBDY$ are constructed in the exterior of an acute-angled triangle ABC. Prove that the intersection of the lines AD and BE lies on the altitude of the triangle from C.
3. Let j and k be integers. Prove that the inequality
$$[(j+k)\alpha]+[(j+k)\beta]\geq[j\alpha]+[j\beta]+[k(\alpha+\beta)]$$
holds for all real numbers α,β if and only if $j=k$.
4. Let $ABCD$ be a square with side 20, and let T_i $(i=1,2,\dots,2000)$ be points in its interior such that no three points from the set $S=\{A,B,C,D\}\cup\{T_1,\dots,T_{2000}\}$ are collinear. Prove that at least one of the triangles with the vertices in S has the area less than $1/10$.

3-rd Grade

1. Let B and C be fixed points, and let A be a variable point such that $\angle BAC$ is fixed. The midpoints of AB and AC are D and E respectively, and F,G are points such that $DF\perp AB$, $EG\perp AC$ and BF and CG are perpendicular to BC. Prove that $BF\cdot CG$ remains constant as A varies.

2. Find all 5-tuples of different four-digit integers with the same initial digit such that the sum of the five numbers is divisible by four of them.

3. A plane intersects a rectangular parallelepiped in a regular hexagon. Prove that the rectangular parallelepiped is a cube.

4. If $n \geq 2$ is an integer, prove the equality

$$[\log_2 n] + [\log_3 n] + \cdots + [\log_n n] = [\sqrt{n}] + [\sqrt[3]{n}] + \cdots + [\sqrt[n]{n}].$$

4-th Grade

1. Let $\mathscr{P}$ be the parabola given by $y^2 = 2px$, and let T_0 be a point on it. Point T_0' is such that the midpoint of the segment T_0T_0' lies on the axis of the parabola. For a variable point T on $\mathscr{P}$, the perpendicular from T_0' to the line T_0T intersects the line through T parallel to the axis of $\mathscr{P}$ at a point T'. Find the locus of T'.

2. A circle is centered on the basis BC of an isosceles triangle ABC and touches the equal sides AB and AC. Let P and Q be points on the sides AB and AC, respectively. Prove that $PB \cdot CQ = \left(\frac{1}{2}BC\right)^2$ if and only if PQ is tangent to the circle.

3. Let $n \geq 3$ positive integers $a_1, \dots, a_n$ be written on a circle so that each of them divides the sum of its two neighbors. Let us denote

$$S_n = \frac{a_n + a_2}{a_1} + \frac{a_1 + a_3}{a_2} + \cdots + \frac{a_{n-2} + a_n}{a_{n-1}} + \frac{a_{n-1} + a_1}{a_n}.$$

Determine the minimum and maximum values of S_n.

4. Let S be the set of all squarefree numbers and n be a natural number. Prove that

$$\sum_{k \in S} \left[\sqrt{\frac{n}{k}}\right] = n.$$

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