5-th Czech–Slovak Match 1999

Bilovec, June 8–11, 1999

1. For arbitrary positive numbers a, b, c, prove the inequality
$$\frac{a}{b+2c}+\frac{b}{c+2a}+\frac{c}{a+2b}\geq 1.$$

2. The altitudes through the vertices A, B, C of an acute-angled triangle ABC meet the opposite sides at D, E, F, respectively. The line through D parallel to EF meets the lines AC and AB at Q and R, respectively. The line EF meets BC at P. Prove that the circumcircle of the triangle PQR passes through the midpoint of BC.

3. Find all natural numbers k for which there exists a set M of ten real numbers such that there are exactly k pairwise non-congruent triangles whose side lengths are three (not necessarily distinct) elements of M.

4. Find all positive integers k for which the following statement is true: If $F(x)$ is a polynomial with integer coefficients satisfying the condition
$$0 \leq F(c) \leq k \quad \text{for each } c \in \{0, 1, \dots, k+1\},$$
then $F(0) = F(1) = \cdots = F(k+1)$.

5. Find all functions $f : (1, \infty) \to \mathbb{R}$ that satisfy
$$f(x) - f(y) = (y - x) f(xy) \quad \text{for all } x, y > 1.$$

6. Prove that for any integer $n \geq 3$, the least common multiple of the numbers $1, 2, \dots, n$ is greater than 2^{n-1}.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com