1-st Czech–Slovak Match 1995

Jevíčko, June 11-12, 1995

1. The sequence $a_1, a_2, \dots$ is defined by $a_1 = 2$, $a_2 = 5$ and
$$a_{n+2} = \left(2-n^2\right)a_{n+1} + \left(2+n^2\right)a_n \quad \text{for each } n \in \mathbb{N}.$$
Decide whether there exist indices p, q and r such that $a_p a_q = a_r$.

2. Find all pairs of functions $f, g : \mathbb{Z} \to \mathbb{Z}$ that satisfy
$$f(g(x)+y) = g(f(y)+x) \quad \text{for all integers } x, y$$
and such that $g(x) = g(y)$ only if $x = y$.

3. Consider all triangles ABC in the cartesian plane whose vertices are at lattice points (i.e. with integer coordinates) and which contain exactly one lattice point (to be denoted P) in its interior. Let the line AP meet BC at E. Determine the maximum possible value of the ratio $\dfrac{AP}{PE}$.

4. For each real number $p > 1$, find the minimum possible value of the sum $x+y$, where the numbers x and y satisfy the equation
$$\left(x+\sqrt{1+x^2}\right)\left(y+\sqrt{1+y^2}\right) = p.$$

5. The diagonals of a convex quadrilateral $ABCD$ are orthogonal and intersect at point E. Prove that the reflections of E in the sides of quadrilateral $ABCD$ lie on a circle.

6. Find all nonnegative integer solutions (x, y) of the equation $y^x - y^p = 1$, where p is a given odd prime number.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
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