# 4-th Czech–Polish–Slovak Match 2004

## Bílovec, June 21–22, 2004

1. Show that real numbers $p$, $q$, $r$ satisfy the condition

$$p^4(q-r)^2+2p^2(q+r)+1=p^4$$

if and only if the quadratic equations $x^2+px+q=0$ and $y^2-py+r=0$ have real roots (not necessarily distinct) which can be labeled by $x_1,x_2$ and $y_1,y_2$, respectively, in such a way that $x_1y_1-x_2y_2=1$.

2. Show that for each natural number $k$ there exist only finitely many triples $(p,q,r)$ of distinct primes for which $p$ divides $qr-k$, $q$ divides $pr-k$, and $r$ divides $pq-k$.

3. A point $P$ in the interior of a cyclic quadrilateral $ABCD$ satisfies $\angle BPC = \angle BAP + \angle PDC$. Denote by $E$, $F$ and $G$ the feet of the perpendiculars from $P$ to the lines $AB$, $AD$ and $DC$, respectively. Show that the triangles $FEG$ and $PBC$ are similar.

4. Solve in the real numbers the system of equations

$$\begin{cases} \dfrac{1}{xy}=\dfrac{x}{z}+1 \\ \dfrac{1}{yz}=\dfrac{y}{x}+1 \\ \dfrac{1}{zx}=\dfrac{z}{y}+1 \end{cases}$$

5. Points $K,L,M$ on the sides $AB,BC,CA$ respectively of a triangle $ABC$ satisfy $\frac{AK}{KB}=\frac{BL}{LC}=\frac{CM}{MA}$. Show that the triangles $ABC$ and $KLM$ have a common orthocenter if and only if $\triangle ABC$ is equilateral.

6. On the table there are $k \geq 3$ heaps of $1,2,\dots,k$ stones. In the first step, we choose any three of the heaps, merge them into a single new heap, and remove 1 stone from this new heap. Thereafter, in the $i$-th step ($i \geq 2$) we merge some three heaps containing more than $i$ stones in total and remove $i$ stones from the new heap. Assume that after a number of steps a single heap of $p$ stones remains on the table. Show that the number $p$ is a perfect square if and only if so are both $2k+2$ and $3k+1$. Find the least $k$ with this property.