3-rd Czech–Polish–Slovak Match 2003

Žilina, June 15–18, 2003

1. Given an integer $n \geq 2$, solve in real numbers the system of equations

$$\begin{array}{rcl} \max\{1,x_1\} & = & x_2 \\ \max\{2,x_2\} & = & 2x_3 \\ \cdots\cdots\cdots & & \cdots \\ \max\{n,x_n\} & = & nx_1. \end{array}$$

2. In an acute-angled triangle ABC the angle at B is greater than 45°. Points D,E,F are the feet of the altitudes from A,B,C respectively, and K is the point on segment AF such that $\angle DKF = \angle KEF$.

   (a) Show that such a point K always exists.

   (b) Prove that $KD^2 = FD^2 + AF \cdot BF$.

3. Numbers p,q,r lie in the interval $(\frac{2}{5},\frac{5}{2})$ and satisfy $pqr = 1$. Prove that there exist two triangles of the same area, one with the sides a,b,c and the other with the sides pa,qb,rc.

4. Point P lies on the median from vertex C of a triangle ABC. Line AP meets BC at X, and line BP meets AC at Y. Prove that if quadrilateral $ABXY$ is cyclic, then triangle ABC is isosceles.

5. Find all natural numbers $n \geq 2$ for which all binomial coefficients

$$\binom{n}{1},\binom{n}{2},\dots,\binom{n}{n-1}$$

   are even numbers.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the condition

$$f(f(x)+y) = 2x + f(f(y)-x) \quad \text{for all } x,y \in \mathbb{R}.$$

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