2-nd Czech–Polish–Slovak Match 2002

Zwardoń, Poland
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1. Let a, b be distinct real numbers and k, m be positive integers $k+m=n\geq 3$, $k\leq 2m$, $m\leq 2k$. Consider sequences $x_1,\dots,x_n$ with the following properties:

   (i) k terms x_i, including x_1, are equal to a;

   (ii) m terms x_i, including x_n, are equal to b;

   (iii) no three consecutive terms are equal.

   Find all possible values of $x_nx_1x_2+x_1x_2x_3+\cdots+x_{n-1}x_nx_1$.

2. A triangle ABC has sides $BC=a$, $CA=b$, $AB=c$ with $a<b<c$ and area S. Determine the largest number u and the least number v such that, for every point P inside $\triangle ABC$, the inequality $u\leq PD+PE+PF\leq v$ holds, where D,E,F are the intersection points of AP, BP, CP with the opposite sides.

3. Let $S=\{1,2,\dots,n\}$, $n\in\mathbb{N}$. Find the number of functions $f:S\to S$ with the property that $x+f(f(f(f(x))))=n+1$ for all $x\in S$?

4. An integer $n>1$ and a prime p are such that n divides $p-1$, and p divides n^3-1. Prove that $4p-3$ is a perfect square.

5. In an acute-angled triangle ABC with circumcenter O, points P and Q are taken on sides AC and BC respectively such that $\frac{AP}{PQ}=\frac{BC}{AB}$ and $\frac{BQ}{PQ}=\frac{AC}{AB}$. Prove that the points O,P,Q,C lie on a circle.

6. Let $n\geq 2$ be a fixed even integer. We consider polynomials of the form

$$P(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+1$$

   with real coefficients, having at least one real roots. Find the least possible value of $a_1^2+a_2^2+\cdots+a_{n-1}^2$.

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