Chinese IMO Team Selection Test 1999

Time: 4.5 hours each day.

First Day

1. Let $x_1, x_2, \ldots, x_n$ be positive reals whose sum equals 1. Find the maximum possible value of $\sum_{i=1}^{n} (x_i^4 - x_i^5)$.

2. Find all prime numbers $p$ with the property that, for all primes $q$, the remainder of $p$ upon division by $q$ is squarefree (i.e. not divisible by any square greater than 1).

3. Find the least $n$ for which there exist $n$ subsets $A_1, A_2, \ldots, A_n$ of set $S = \{1, 2, \ldots, 15\}$ satisfying:
   (i) $|A_i| = 7$ for all $i$;
   (ii) $|A_i \cap A_j| \leq 3$ for any two distinct $i, j$;
   (iii) for any 3-element subset $M \subset S$ there is an $A_k$ containing $M$.

Second Day

4. Let a circle touch the sides $AB, BC$ of a convex quadrilateral $ABCD$ at $G$ and $H$ and intersect $AC$ at $E$ and $F$. Find the condition $ABCD$ must satisfy in order to exist a circle passing through $E, F$ and touching $DA, DC$.

5. Let $m$ be an even positive integer.
   (a) Show that there exist integers $x_1, x_2, \ldots, x_{2m}$ such that $x_i x_{i+m} = x_{i+1} x_{i+m-1} + 1$ for $i = 1, 2, \ldots, m - 1$.
   (b) Prove that, if $x_1, x_2, \ldots, x_{2m}$ satisfy (a), one can construct an infinite sequence $(y_k)_{k \in \mathbb{Z}}$ of integers such that $y_i = x_i$ for $i = 1, \ldots, 2m$ and $y_k y_{k+m} = y_{k+1} y_{k+m-1} + 1$ for all integers $k$.

6. For all permutations $\tau = (x_1, \ldots, x_{10})$ of numbers $1, 2, \ldots, 10$, define
   $$S(\tau) = \sum_{i=1}^{10} |2x_i - 3x_{i+1}|$$
   (where $x_{11} = x_1$). Find the maximum and minimum values of $S(\tau)$ and all the permutations $\tau$ for which those are attained.