Chinese IMO Team Selection Test 1998

First Day

1. Find the positive integer k such that:

   (i) There are no integers $n > 0$ and $0 \leq j \leq n-k-1$ for which $\binom{n}{j}, \binom{n}{j+1}, \dots, \binom{n}{j+k-1}$ form an arithmetic progression;

   (ii) There exist integers $n > 0$ and $0 \leq j \leq n-k+2$ for which $\binom{n}{j}, \binom{n}{j+1}, \dots, \binom{n}{j+k-1}$ form an arithmetic progression.

   For this k, find all n for which there is j satisfying (ii).

2. On a football tournament with n teams participating, every two teams played exactly one match. A team is awarded 3 points for a victory, 1 point for a draw, and 0 points for a defeat. When the tournament was over, the third from the bottom team had a smaller score than the teams above it, and a greater score than the teams below it. However, this team has more victories than the teams above it, and less than those below it. Find the least n for which this is possible.

3. For a fixed $\theta \in (0, \frac{\pi}{2})$, find the smallest positive constant a with the following properties:

   (i) $\frac{\sqrt{a}}{\cos\theta} + \frac{\sqrt{a}}{\sin\theta} > 1$.

   (ii) There exists x with $1 - \frac{\sqrt{a}}{\sin\theta} \leq x \leq \frac{\sqrt{a}}{\cos\theta}$ such that

$$\left((1-x)\sin\theta - \sqrt{a - x^2\cos^2\theta}\right)^2 + \left(x\cos\theta - \sqrt{a-(1-x)^2\sin^2\theta}\right)^2 \leq a.$$

Second Day

4. In an acute triangle ABC, H is the orthocenter, O the circumcenter, and I the incenter. Given that $\angle C > \angle B > \angle A$, prove that I lies within $\triangle BOH$.

5. On a line l in the plane are given $n \geq 3$ distinct points $P_1, P_2, \dots, P_n$. For $i = 1, \dots, n$, let d_i denote the product of the distances from P_i to the other $n-1$ points. Let Q be a point in the plane outside l and let $C_i = |QP_i|$. for $i = 1, \dots, n$. Determine
$$S_n = \sum_{i=1}^{n} (-1)^{n-i} \frac{C_i^2}{d_i}.$$

6. For any $h = 2^r$ (r is a non-negative integer), find all $k \in \mathbb{N}$ which satisfy the following condition: There exist natural numbers $m > 1$, n with m odd such that $k \mid m^h - 1$ and $m \mid n^{\frac{m^h-1}{k}} + 1$.

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