Chinese IMO Team Selection Test 1998

First Day

1. Find the positive integer \( k \) such that:
   
   (i) There are no integers \( n > 0 \) and \( 0 \leq j \leq n - k - 1 \) for which \( \binom{n}{j}, \binom{n}{j+1}, \ldots, \binom{n}{j+k-1} \) form an arithmetic progression;
   
   (ii) There exist integers \( n > 0 \) and \( 0 \leq j \leq n - k + 2 \) for which \( \binom{n}{j}, \binom{n}{j+1}, \ldots, \binom{n}{j+k-1} \) form an arithmetic progression.

   For this \( k \), find all \( n \) for which there is \( j \) satisfying (ii).

2. On a football tournament with \( n \) teams participating, every two teams played exactly one match. A team is awarded 3 points for a victory, 1 point for a draw, and 0 points for a defeat. When the tournament was over, the third from the bottom team had a smaller score than the teams above it, and a greater score than the teams below it. However, this team has more victories than the teams above it, and less than those below it. Find the least \( n \) for which this is possible.

3. For a fixed \( \theta \in (0, \frac{\pi}{2}) \), find the smallest positive constant \( a \) with the following properties:
   
   (i) \( \frac{\sqrt{a}}{\cos \theta} + \frac{\sqrt{a}}{\sin \theta} > 1 \).
   
   (ii) There exists \( x \) with \( 1 - \frac{\sqrt{a}}{\sin \theta} \leq x \leq \frac{\sqrt{a}}{\cos \theta} \) such that
   
   \[ \left( (1-x)\sin \theta - \sqrt{a-x^2\cos^2 \theta} \right)^2 + \left( x\cos \theta - \sqrt{a-(1-x)^2\sin^2 \theta} \right)^2 \leq a. \]

Second Day

4. In an acute triangle \( ABC \), \( H \) is the orthocenter, \( O \) the circumcenter, and \( I \) the incenter. Given that \( \angle C > \angle B > \angle A \), prove that \( I \) lies within \( \triangle BOH \).

5. On a line \( l \) in the plane are given \( n \geq 3 \) distinct points \( P_1, P_2, \ldots, P_n \). For \( i = 1, \ldots, n \), let \( d_i \) denote the product of the distances from \( P_i \) to the other \( n - 1 \) points. Let \( Q \) be a point in the plane outside \( l \) and let \( C_i = |QP_i| \), for \( i = 1, \ldots, n \). Determine
   
   \[ S_n = \sum_{i=1}^{n} (-1)^{n-i} \frac{C_i^2}{d_i}. \]

6. For any \( h = 2^r \) (\( r \) is a non-negative integer), find all \( k \in \mathbb{N} \) which satisfy the following condition: There exist natural numbers \( m > 1, n \) with \( m \) odd such that \( k | m^h - 1 \) and \( m | n^{2^{h-1}} + 1 \).