Chinese IMO Team Selection Test 1996

Time: 4.5 hours each day.

First Day

1. The circle with side BC of a triangle ABC as the diameter cuts AB and AC at D and E respectively. Let F and G be the feet of the perpendiculars from D and E to BC, respectively. The lines DG and EF intersect at M. Prove that $AM \perp BC$.

2. Let S be the set of functions $f : \mathbb{N} \to \mathbb{R}$ satisfying the following conditions:

   (i) $f(1) = 2$

   (ii) $f(n+1) \geq f(n) \geq \frac{n}{n+1} f(2n)$ for $n = 1, 2, \ldots$

   Find the least integer M such that $f(n) < M$ holds for any $f \in S$ and any $n \in \mathbb{N}$.

3. Find the smallest natural number n such that every n-element subset of $M = \{2, 3, \ldots, 1000\}$ contains 3 pairwise disjoint 4-element subsets S, T, U such that:

   (i) For any two elements in the same subset, one divides the other.

   (ii) For any $s \in S$ and $t \in T$, $\gcd(s,t) = 1$.

   (iii) For any $s \in S$ and $u \in U$, $\gcd(s,u) > 1$.

Second Day

4. Three countries A, B, C, each with 9 representatives, take part in a championship. Every match is played between two competitors from two different countries, and the winner plays the next match with a representative of the third country, while the loser is eliminated. When all representatives of one country are eliminated, the competition continues between the remaining two countries until all representatives of another country are eliminated. The remaining country is the champion.

   (i) At least how many games did the champion country win?

   (ii) If the champion won 11 matches, at least how many matches were played?

5. Let $\alpha_1, \ldots, \alpha_n,\ \beta_1, \ldots, \beta_n$ $(n \geq 4)$ be real numbers with $\sum_{i=1}^{n} \alpha_i^2 < 1$ and $\sum_{i=1}^{n} \beta_i^2 < 1$. Let

$$A^2 = 1 - \sum_{i=1}^{n} \alpha_i^2, \quad B^2 = 1 - \sum_{i=1}^{n} \beta_i^2, \quad W = \frac{1}{2}\left(1 - \sum_{i=1}^{n} \alpha_i \beta_i\right)^2.$$

   Find all real numbers λ such that all roots of the equation $x^n + \lambda(x^{n-1} + \cdots + x^3 + Wx^2 + ABx + 1) = 0$ are real.

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6. Do there exist non-zero complex numbers a, b, c and natural number h with the property that

$$|ka+lb+mc| > \frac{1}{h}$$

holds whenever k, l, m are integers with $|k|+|l|+|m| \geq 1996$?

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