Chinese IMO Team Selection Test 1992

Time: 4.5 hours each day.

First Day – April 17

1. On a math competition 16 contestants take part. Each problem has four possible answers, and a contestant must choose one. After the examination, it has been found that any two contestants have at most one answer in common. How many problems at most are there in the examination? Justify your conclusion.
2. Let $n \geq 2$ be a given integer. Find the smallest positive number λ such that for any $a_1, a_2, \dots, a_n \geq 0$ and any $0 \leq b_1, b_2, \dots, b_n \leq 1/2$ with $a_1 + \cdots + a_n = b_1 + \cdots + b_n = 1$ it holds that
$$a_1 a_2 \cdots a_n \leq \lambda (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n).$$
3. Let p be a prime number. Show that there exists an integer x such that $p \mid x^2 - x + 3$ if and only if there exists an integer y such that $p \mid y^2 - y + 25$.

Second Day – April 18

4. Let ABC be a triangle with $AB = \sqrt{7}$, $BC = \sqrt{13}$, $CA = \sqrt{19}$. Circles α, β, γ with centers at A, B, C and radii $1/3$, $2/3$ and 1 respectively are constructed in the plane of the triangle. Show that there are three points $A' \in \alpha$, $B' \in \beta$, $C' \in \gamma$ such that triangles ABC and $A'B'C'$ are congruent.
5. A square of side $3n+1$ ($n \in \mathbb{N}$) is divided into unit squares. Prove that, if one of the unit squares is cut off, then the remainder of the big square can be tiled with $L-$trominoes.
6. Let n, T be given integers ≥ 2. Find all positive integers a such that
$$\sum_{k=1}^{n} \frac{ka + a^2/4}{s_k} < T^2 \sum_{k=1}^{n} \frac{1}{a_k}$$
holds for any positive integers $a_1, a_2, \dots, a_n$, where $s_k = a_1 + a_2 + \cdots + a_k$.

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