Chinese IMO Team Selection Test 1991

Time: 4.5 hours each day.

First Day – April 17

1. Let be given the polynomial $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ $(n \geq 2)$ which has real roots $b_1, b_2, \dots, b_n$. Prove that for every $x > \max\{b_1, \dots, b_n\}$ it holds that

$$f(x+1) \geq \frac{2n^2}{\frac{1}{x-b_1} + \frac{1}{x-b_2} + \cdots + \frac{1}{x-b_n}}.$$

2. Let be given a circle. For each $i = 1, 2, \dots, 1991$, n_i points are arbitrarily selected on the circle and i is written at each of them. Find a necessary and sufficient condition on $n_1, n_2, \dots, n_{1991}$ such that one can connect the points by some chords satisfying:

   (i) no two chords have a common point;

   (ii) at the endpoints of every chord are written different numbers;

   (iii) each of the selected points is an endpoint of a chord?

3. Five points are given in the plane such that no three are collinear and no four are concyclic. We call a circle *good* if it passes through exactly three of the given points and contains exactly one of the remaining two points inside. Find all possible values of the number of good circles.

Second Day – April 18

4. Five points $A_1, A_2, \dots, A_5$ are given in this order on a unit circle. Let P be a point inside the circle. For $i = 1, 2, \dots, 5$, lines $A_i A_{i+2}$ and PA_{i+1} intersect at Q_i, where $A_6 = A_1$ and $A_7 = A_2$. Given that $OQ_i = d_i$ for $i = 1, 2, \dots, 5$, determine the product

$$A_1Q_1 \cdot A_2Q_2 \cdots A_5Q_5.$$

5. The function f is defined on the set of nonnegative integers by $f(0) = 0$, $f(1) = 1$ and

$$f(n+2) = 23f(n+1) + f(n), \qquad n = 0, 1, 2, \dots$$

   Prove that for any $m \in \mathbb{N}$ there is a $d \in \mathbb{N}$ such that $m \mid f(f(n))$ if and only if $d \mid n$.

6. Every edge of a convex polyhedron is colored red or blue. We call an angle of a face *singular* if its two rays are of different colors. For any vertex A of the polyhedron, let S_A denote the number of singular angles at A. Prove that there exist two vertices B and C such that $S_B + S_C \leq 4$.

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