Chinese IMO Team Selection Tests 1989

First Test

1. A triangle with sides $\frac{3}{2}$, $\frac{\sqrt{5}}{2}$, $\sqrt{2}$ is folded along a variable line perpendicular to the side of length $\frac{3}{2}$. What is the maximum area of the overlapping region?

2. Consider the sequence $v_0 = 0$, $v_1 = 1$, $v_{n+1} = 8v_n - v_{n-1}$ for $n = 1, 2, \dots$. Prove that no term of this sequence is of the form $3^\alpha 5^\beta$ for positive integers α, β.

3. Find the largest integer n for which all nonzero roots of $(z+1)^n = z^n + 1$ have the modulus 1.

4. Squares $ABEF$, $BCGH$, $CAIJ$ are constructed outwards on the sides of triangle ABC. Let $AH \cap BJ = P_1$, $BJ \cap CF = Q_1$, $CF \cap AH = R_1$, $AG \cap CE = P_2$, $BI \cap AG = Q_2$, $CE \cap BI = R_2$. Prove that the triangles $P_1Q_1R_1$ and $P_2Q_2R_2$ are congruent.

Second Test

1. Does there exist a function $f : \mathbb{N} \mapsto \mathbb{N}$ such that $f^{1989}(n) = 2n$ for all natural n? (f^k denotes f iterated k times.)

2. In a triangle ABC, AD is the altitude. If $BC + AD - AB - AC = 0$, find the possible values of $\angle BAC$.

3. There are 1989 non-overlapping disks arbitrarily placed on the table. What is the least number of colors with which one can always paint the circles, each in one color, so that any two tangent circles are of different colors?

4. For a natural number n, $P(n)$ denotes the number of partitiones of n into a sum of positive integers (the order of the summands is irrelevant). For instance, $P(4) = 5$. The number of distinct summands in a partition is called its *dispersion*. Let $q(n)$ be the sum of the dispersions over all partitions of n. Prove that:

   (a) $q(n) = 1 + P(1) + P(2) + \cdots + P(n-1)$;

   (b) $1 + P(1) + P(2) + \cdots + P(n-1) \le \sqrt{2n}P(n)$.

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