Chinese IMO Team Selection Test 2003

Time: 4.5 hours each day.

First Day – March 31

1. In an acute triangle ABC, the angle bisector of $\angle A$ meets side BC at D. Let E and F be the feet of perpendiculars from D to AC and AB respectively. Lines BE and CF intersect at H, and the circumcircle of $\triangle AFH$ meets BE at H and G. Show that the triangle with side lengths BG, GE, BF is right-angled.

2. Find the subset A of $\{0,1,2,\dots,29\}$ of the greatest possible cardinality with the following property: for any integer k and any $a,b \in A$ (not necessarily distinct), the number $a+b+30k$ is not a product of two consecutive integers.

3. For any $\alpha = (a_1, a_2, \dots, a_n)$ and $\beta = (b_1, b_2, \dots, b_n)$ from $\mathbb{R}^n$, define

$$\gamma(\alpha,\beta) = (|a_1-b_1|, |a_2-b_2|, \dots, |a_n-b_n|)\,.$$

For a finite subset A of $\mathbb{R}^n$, let $D(A) = \{\gamma(\alpha,\beta) \mid \alpha,\beta \in A\}$. Show that $|D(A)| \geq |A|$.

Second Day – April 1

4. Find all functions $f : \mathbb{N} \to \mathbb{R}$ satisfying:

   (i) $f(n+1) \geq f(n)$ for all $n \geq 1$;

   (ii) $f(mn) = f(m)f(n)$ for any coprime m and n.

5. Consider $A = \{1,2,\dots,2002\}$ and $M = \{1001, 2003, 3005\}$. We say that a nonempty subset B of A is *M-free* if the sum of any two elements of B is not in M. If $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$ and both A_1, A_2 are M-free, we say that the ordered pair (A_1, A_2) is an *M-partition* of A. Find the number of M-partitions of A.

6. The sequence (x_n) satisfies $x_0 = 0$, $x_2 = x_1\sqrt[3]{2}$, $x_3 \in \mathbb{N}$ and

$$x_{n+1} = \frac{1}{\sqrt[3]{4}}x_n + \sqrt[3]{4}x_{n-1} + \frac{1}{2}x_{n-2} \qquad \text{for all } n \geq 2.$$

Determine the minimum number of integer terms that the sequence must have.

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