Chinese IMO Team Selection Test 2002

Time: 4.5 hours each day.

First Day

1. Let $ABCD$ be a convex quadrilateral with no two sides parallel. Lines AB and CD meet at point E, lines BC and AD meet at point F, and diagonals AC and BD meet at P. If O is the foot of perpendicular from P to EF, prove that $\angle BOC = \angle AOD$.

2. Let the sequence a_n be defined by $a_0 = 1/4$ and $a_n = (1 + a_{n-1})^2/4$, and let
$$A_k = \frac{x_k - k}{\left(x_k + x_{k+1} + \cdots + x_{2002} + \frac{k(k+1)}{2} - 1\right)^2}$$
for $k = 1, 2, \dots, 2002$. Find the smallest $\lambda \in \mathbb{R}$ for which
$$A_1 + A_2 + \cdots + A_{2002} \le \lambda a_{2002}.$$

3. Seventeen football fans bought tickets for seventeen matches on the World Cup. It turned out that:
   - (i) Each person bought at most one ticket for each match;
   - (ii) For any two fans, there is at most one match for which they both bought a ticket;
   - (iii) Exactly one of the fans bought six tickets.

   What is the maximum possible number of tickets bought by these fans? Justify your answer.

Second Day

4. (a) Find all positive integers $n \ge 2$ for which there exist n integers $x_1, \dots, x_n$ such that $\{|x_i - x_j| \mid 1 \le i < j \le n\} = \{1, \dots, \frac{n(n+1)}{2}\}$.

   (b) Let $A = \{1, 2, \dots, 6\}$ and $B = \{7, 8, \dots, n\}$. Find the smallest n for which there exist five-element sets $A_1, A_2, \dots, A_{20}$ with the following properties: for each i, j with $1 \le i < j \le 20$, $|A_i \cap A| = 3$, $|A_i \cap B| = 2$, $|A_i \cap A_j| \le 2$.

5. Let S be the set of all negative integers. Given an integer k, find all functions $f : S \to \mathbb{Z}$ such that
$$f(n)f(n+1) = (f(n) + n - k)^2 \qquad \text{for all } n \le -2.$$

6. Define
$$\begin{aligned} f(x,y,z) = \quad & -2(x^3 + y^3 + z^3) + 3(x_1^2(x_2 + x_3) + \\ & + x_2^2(x_3 + x_1) + x_3^2(x_1 + x_2)) - 12x_1x_2x_3. \end{aligned}$$
Find the minimum value of
$$g(r,s,t) = \max_{t \le x_3 \le t+2} |f(r, r+2, x_3) + s|.$$

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