Chinese IMO Team Selection Test 2001

Time: 4.5 hours each day.

First Day

1. A convex quadrilateral $ABCD$ is given in the plane. Suppose there exist points E,F inside the quadrilateral such that
$$AE = BE,\quad CE = DE,\quad \angle AEB = \angle CED;\qquad \text{and}$$
$$AF = DF,\quad BF = CF,\quad \angle AFD = \angle BFC.$$
Prove that $\angle AFD + \angle AEB = \pi$.

2. Let a,b be integers with $b > a > 1$ such that a does not divide b, and let $(b_n)_{n=1}^{\infty}$ be a sequence of natural numbers such that $b_{n+1} \geq 2b_n$ for all n. Does there necessarily exist a sequence $(a_n)_{n=1}^{\infty}$ of natural numbers such that $a_{n+1} - a_n \in \{a,b\}$ for all n and $a_m + a_l \notin (b_n)_{n=1}^{\infty}$ for any indices m,l?

3. Let $k \geq 1$ be a given integer. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x^k + f(y)) = y + f(x)^k \qquad \text{for all } x,y \in \mathbb{R}.$$

Second Day

4. Let $n \geq 3$ be an integer and let $0 < x_1 < x_2 < \cdots < x_{n+2}$ be real numbers. Find the minimum possible value of
$$\frac{\left(\sum_{i=1}^{n} \frac{x_{i+1}}{x_i}\right)\left(\sum_{j=1}^{n} \frac{x_{j+2}}{x_{j+1}}\right)}{\left(\sum_{k=1}^{n} \frac{x_{k+1}x_{k+2}}{x_{k+1}^2 + x_k x_{k+2}}\right)\left(\sum_{l=1}^{n} \frac{x_{l+1}^2 + x_l x_{l+2}}{x_l x_{l+1}}\right)},$$
and determine when this minimum value is attained.

5. Let D be an arbitrary point on the side BC of an equilateral triangle ABC. Let O_1, I_1 be the circumcenter and incenter of $\triangle ABD$ and O_2, I_2 be the circumcenter and incenter of $\triangle ACD$, respectively. Lines O_1I_1 and O_2I_2 meet at a point P. Find the locus of P as D moves along the segment BC.

6. Find positive real numbers a,b,c for which $F = \max\limits_{1 \leq x \leq 3} |x^3 - ax^2 - bx - c|$ is minimum possible.

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