31-st Canadian Mathematical Olympiad 1999

1. Solve the equation $4x^2 - 40[x] + 51 = 0$ in real numbers.

2. Let ABC be an equilateral triangle of altitude 1. A circle with radius 1 and center on the same side of AB as C rolls along the segment AB. Prove that the arc of the circle that is inside the triangle always has the same length.

3. Determine all positive integers n with the property that $n = d(n)^2$, where $d(n)$ denotes the number of positive divisors of n.

4. Let $a_1, a_2, \dots, a_8$ be eight distinct integers from $\{1, 2, \dots, 17\}$. Show that there is an integer $k > 0$ such that the equation $a_i - a_j = k$ has at least three different solutions. Also, find seven distinct integers $a_1, \dots, a_7$ from $\{1, 2, \dots, 17\}$ such that the equation $a_i - a_j = k$ has at most two distinct solutions for any $k > 0$.

5. Let x, y, z be non-negative real numbers satisfying $x + y + z = 1$. Prove the inequality
$$x^2y + y^2z + z^2x \leq \frac{4}{27},$$
and find when equality occurs.

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