

28-th Canadian Mathematical Olympiad 1996

1. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute

$$\frac{1 + \alpha}{1 - \alpha} + \frac{1 + \beta}{1 - \beta} + \frac{1 + \gamma}{1 - \gamma}.$$

2. Find all real solutions to the following system of equations:

$$\frac{4x^2}{1 + 4x^2} = y, \quad \frac{4y^2}{1 + 4y^2} = z, \quad \frac{4z^2}{1 + 4z^2} = x.$$

3. Let $f(n)$ denote the number of permutations a_1, a_2, \dots, a_n of $1, 2, \dots, n$ such that

- (i) $a_1 = 1$;
- (ii) $|a_i - a_{i+1}| \leq 2$ for $i = 1, \dots, n - 1$.

Determine whether $f(1996)$ is divisible by 3.

4. Let ABC be an isosceles triangle with $AB = AC$, and let the bisector of $\angle B$ meet AC at D . Suppose that $BC = BD + AD$. Determine $\angle A$.

5. Let r_1, r_2, \dots, r_m be positive rational numbers with the sum 1. Define the function f by $f(n) = n - \sum_{k=1}^m [r_k n]$ for each $n \in \mathbb{N}$. Determine the minimum and maximum values of $f(n)$.